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# Determination of formal CR mappings by a finite jet

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**Abstract**

We prove that if $M$ is a connected real-analytic holomorphically nondegenerate hypersurface in $\mathbb{C}^{n+1}$, then for any point $p \in M$ there exists an integer $k$ such that any two germs at $p$ of local biholomorphic mappings $H_1, H_2 : (\mathbb{C}^{n+1}, p) \to (\mathbb{C}^{n+1}, p)$ that send $M$ into itself and whose $k$-jets agree at $p$ are identical.

The above is a special case of a more general theorem stated for formal hypersurfaces that gives a finite jet determination result for the class of formal mappings whose Jacobian determinant does not vanish identically.


*MSC:* 32H02; 32V40

*Keywords:* CR mapping; Finite jet determination

## 1. Introduction and statements of main results

In this paper, we prove the following result which provides a positive answer to a conjecture of Baouendi–Ebenfelt–Rothschild, essentially going back to [2] (see also, e.g., [7] and [17]):

**Theorem 1.1.** *Let $M \subseteq \mathbb{C}^{n+1}$ be a connected real-analytic holomorphically nondegenerate hypersurface, where $n \geqslant 1$. Then, for every $p \in M$, there is a nonnegative integer $N = N(M, p)$ such that the germs at $p$ of local real-analytic CR automorphisms of $M$ are uniquely determined by their $N$-jets at $p$.*

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Recall that an $N$-jet of a function $h$ at a point $p$, denoted $j_p^N(h)$, is the collection of derivatives of $h$ at $p$ up to order $N$. Also, recall that a submanifold in complex space is called holomorphically nondegenerate at a point if it does not have a nontrivial germ of a holomorphic vector field (with holomorphic coefficients) tangent to the manifold in a neighborhood of that point (this notion was introduced by Stanton [23]).

This theorem can be seen as a local CR version for real-analytic hypersurfaces of $\mathbb{C}^{n+1}$ of the classical uniqueness theorem of H. Cartan stating that a holomorphic automorphism of bounded domains in $\mathbb{C}^{n+1}$ is uniquely determined by its 1-jet at any point in the domain.

The study of finite jet-determination for hypersurfaces has a long history. Poincaré in 1907 [21] pointed out the importance of studying holomorphic maps that take a piece of one hypersurface into another. Finite jet determination for Levi-nondegenerate hypersurfaces follows from the works of E. Cartan [9,10] (for hypersurfaces in $\mathbb{C}^2$), and the works of Tanaka [24] and Chern and Moser [11] (for hypersurfaces in higher dimensions). They proved that if the hypersurface $M$ is Levi-nondegenerate at $p \in M$, then any formal invertible map taking $(M, p)$ into itself is uniquely determined by its 2-jet at $p$. If the hypersurface $M$ is more degenerate at the point $p$, we may need more derivatives at $p$ to determine the mapping, so the question arises whether the automorphism can be determined by a finite number of its derivatives. Baouendi, Ebenfelt and Rothschild [2] observed that holomorphic nondegeneracy is necessary in order to have finite jet determination and showed finite jet determination for the subclass of hypersurfaces that are, so-called, finitely nondegenerate see [6] or [3]. They generalized this result to hypersurfaces that are essentially finite at $p$ [4]. For the case when $M$ is of finite type at $p$, Theorem 1.1 was proved by Baouendi, Mir and Rothschild [7]. The first general result when $M$ is of infinite type at $p$ is due to Ebenfelt, Lamel and Zaitsev in [13], namely that Theorem 1.1 holds for hypersurfaces in $\mathbb{C}^2$. Using their techniques Putinar [22] obtained finite jet determination for a class of hypersurfaces of infinite type in higher dimensions. Lamel and Mir [17] showed finite jet determination for the class of hypersurfaces that are generically Levi-nondegenerate.

Theorem 1.1 follows from the following more general theorem, where we consider formal manifolds and the class of formal mappings whose Jacobian determinant does not vanish identically. See Section 2 for precise definitions.

**Theorem 1.2.** *Let $(M, p)$ and $(M', p')$ be formal real hypersurfaces in $\mathbb{C}^{n+1}$ with $(M', p')$ holomorphically nondegenerate. Assume $H_0 : (\mathbb{C}^{n+1}, p) \to (\mathbb{C}^{n+1}, p')$ is a formal mapping sending $M$ into $M'$ whose Jacobian determinant does not vanish identically. Then, there is an integer $N \geqslant 0$ such that if $H$ is any formal mapping sending $(M, p)$ into $(M', p')$ with $j_p^N(H) = j_p^N(H_0)$, then $H \equiv H_0$.*

Finite jet determination can also be studied for maps between manifolds of higher codimension. In the case where the source manifold $M$ is of finite type at $p$, we generalize Theorem 1.2 to generic submanifolds in $\mathbb{C}^{n+d}$ of codimension $d$ (the notions *finite type* and *generic* are defined in Section 2):

**Theorem 1.3.** *Let $(M, p)$ and $(M', p')$ be formal generic manifolds in $\mathbb{C}^{n+d}$ of real codimension $d$ with $(M', p')$ holomorphically nondegenerate and $M$ of finite type at $p$. Assume $H_0 : (\mathbb{C}^{n+d}, p) \to (\mathbb{C}^{n+d}, p')$ is a formal mapping sending $M$ into $M'$ whose Jacobian determinant does not vanish identically. Then, there is an integer $N \geqslant 0$ such that if $H$ is any formal mapping sending $(M, p)$ into $(M', p')$ with $j_p^N(H) = j_p^N(H_0)$, then $H \equiv H_0$.*

The theorem generalizes the corresponding theorem in [7], where in addition the mappings are assumed to be "not totally degenerate".

Under the conditions of Theorem 1.3, we also get an explicit expression for $N$, a bound on the jet order, depending on $H_0$, $M$ and $M'$. As a consequence, we obtain the following result for real-analytic manifolds, where we allow the base point to vary:

**Theorem 1.4.** *Let $M$ and $M'$ be connected real-analytic generic manifolds in $\mathbb{C}^{n+d}$ of real codimension $d$. Assume $H_0$ is a holomorphic mapping defined on an open set $U \subseteq \mathbb{C}^{n+d}$ such that $H_0(M \cap U) \subseteq M'$, $M'$ is holomorphically nondegenerate, and $M$ is of finite type for all points in $M \cap U$. Then, there is a nonnegative integer $N(p)$ depending upper-semicontinuously on $p \in M \cap U$ such that if $H$ is a formal mapping taking $(M, p)$ into $(M', H_0(p))$ with $j_p^{N(p)}(H) = j_p^{N(p)}(H_0)$, then $H = H_0$ as a formal power series at $p$.*

For the class of invertible mappings, the integer $N(p)$ does not depend on the mapping $H_0$. Hence, we have the following result:

**Corollary 1.5.** *Let $M$ be a connected real-analytic generic manifold that is holomorphically nondegenerate and of finite type at all of its points. Then there is a nonnegative integer $N(p)$ depending upper-semicontinuously on the point $p \in M$ such that for any real-analytic generic manifold $M'$ of the same dimension as $M$ and any pairs $H_1$, $H_2$ of formal invertible mappings taking $(M, p)$ into $M'$ with $j_p^{N(p)}(H_1) = j_p^{N(p)}(H_2)$, we have $H_1 \equiv H_2$.*

In their recent paper [16], Lamel and Mir prove a result similar to Corollary 1.5 for a more restrictive class of manifolds, but for a bigger class of CR-mappings—namely the class of finite mappings. In Theorem 1.4, the manifolds and the mappings are allowed to be more degenerate than in their theorem, but the integer $N(p)$ in Theorem 1.4 depends a priori on $H_0$ and $M'$, so in particular the theorem does not hold for arbitrary pairs.

Finite jet determination has also been studied for maps between hypersurfaces of different dimensions, e.g., [15], and for smooth CR-mappings between smooth CR-manifolds see [12]. The second part of this paper uses the techniques introduced by Ebenfelt in [12] and further refined in [13], mentioned above. The calculations used to establish finite jet determination is often very similar to those used to get convergence of formal mappings. Apart from the paper [7], which also deals with convergence, there is an earlier work by Mir [19] that has given some of the inspiration for the first part of this paper.

The paper is organized as follows. We start by introducing basic definitions and notations in Section 2. In Section 3, we establish some finite jet determination results for systems of formal equations that is used later on. In Section 4, we introduce some universal polynomials that are used to establish some mapping identities. These identities will be used in Section 5 to establish finite jet determination along the first Segre variety, and in Section 6, along the higher iterated Segre varieties. We then prove Theorems 1.3 and 1.4, where the source manifold is of finite type (in the sense of Kohn and Bloom–Graham). Finally, in Section 7 we use the techniques from [12] and [13] together with our partial jet determination result from Section 5 to prove Theorem 1.2 in the case when the source hypersurface is of infinite type. Apart from using some results from earlier sections, this section is completely independent and uses different techniques than the rest of the paper.

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## 2. Preliminaries

Let $(M, p)$ be a formal generic manifold in $\mathbb{C}^{n+d}$ of real codimension $d$. This means that $(M, p)$ is generated by a vector-valued formal power series $\rho(Z, \bar{Z}) = (\rho^1(Z, \bar{Z}), \ldots, \rho^d(Z, \bar{Z}))$, where $\rho^j(Z, \bar{Z}) \in \mathbb{C}[\![Z - p, \bar{Z} - \bar{p}]\!]$ for $1 \leqslant j \leqslant d$ and satisfying $\partial\rho^1 \wedge \cdots \wedge \partial\rho^d(p, \bar{p}) \neq 0$ and the reality condition $\bar{\rho}(\bar{Z}, Z) \equiv \rho(Z, \bar{Z})$, where $\bar{\rho}$ is obtained from $\rho$ by taking the complex conjugate of all the coefficients. $\rho$ is called the defining function of the manifold, and we say that the manifold is defined by the equation $\rho(Z, \bar{Z}) = 0$. If $\rho(Z, \bar{Z})$ converges, then the equation $\rho(Z, \bar{Z}) = 0$ defines a (generic) real-analytic submanifold in $\mathbb{C}^{n+d}$ of real codimension $d$ in a neighborhood of the point $p$. For more details about formal generic manifolds, see e.g. [4] or [7].

After a formal change of coordinates, we can (without loss of generality) assume that the manifold is given by normal coordinates at $p$. This means that the point $p$ is at the origin and that the manifold $(M, 0)$ is defined formally by

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w), \tag{2.1}$$

where $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ and $\phi(z, \bar{z}, s)$ is a real vector-valued power series ($z \in \mathbb{C}^n$ and $s \in \mathbb{R}^d$) with $d$ components satisfying $\phi(0, \chi, s) \equiv \phi(z, 0, s) \equiv 0$. By solving for $w$ in (2.1) using the (formal version of the) implicit function theorem, we write this equivalently as

$$w = Q(z, \bar{z}, \bar{w}), \tag{2.2}$$

where $Q$ is a vector-valued complex power series with $d$ components satisfying

$$Q(z, 0, \tau) \equiv \tau \quad \text{and} \quad Q(0, \chi, \tau) \equiv \tau. \tag{2.3}$$

The requirement that $\phi$ is a real-valued power series is equivalent to the reality condition

$$Q\big(z, \chi, \bar{Q}(\chi, z, w)\big) \equiv w. \tag{2.4}$$

We say, following Stanton [23], that a formal manifold $M$ is *holomorphically nondegenerate* if it does not have a nontrivial formal holomorphic vector field, which is tangent to $M$. A formal holomorphic vector field is a vector field of the form $\sum_{j=1}^{n+d} a^j(Z)\frac{\partial}{\partial Z_j}$, where the coefficients $a^j(Z)$ are formal power series in $Z$. If the manifold is given in normal coordinates, then holomorphic nondegeneracy is equivalent (see e.g. [7, Lemma 13.1]) to the condition that there exist $n$ pairs $(\alpha^1, r_1), \ldots, (\alpha^n, r_n)$, where, for each $j$, $\alpha^j$ is a multi-index and $r_j \in \{1, 2, \ldots, d\}$ satisfying

$$\det\left(\left(\frac{\partial^{|\alpha^j|+1} Q^{r_j}}{\partial z^{\alpha^j} \chi_k}(0, \chi, \tau)\right)_{j,k}\right) \not\equiv 0.$$

Recall that a manifold $(M, p)$ is of *finite type at p* (in the sense of Kohn [14] and Bloom and Graham [8]) if the Lie algebra generated by the (formal) $(0, 1)$- and $(1, 0)$-vector fields span the tangent space of $M$ at $p$. In the case of hypersurfaces, then for any choice of normal coordinates for $M$ at $p$, we have that $(M, p)$ is of infinite type at $p$ if and only if

$$Q(z, \chi, 0) \equiv 0. \tag{2.5}$$

We let $(M, p)$ and $(M', p')$ be formal generic manifolds in $\mathbb{C}^{n+d}$ of real codimension $d$, with defining functions $\rho$ and $\rho'$ respectively. We say that a formal mapping $H(Z) \in \mathbb{C}^{n+d}[\![Z-p]\!]$ maps $(M, p)$ into $(M', p')$ if the constant term $H(p) = p'$ and there exists a matrix valued power series $a(Z, \bar{Z}) \in \mathbb{C}^{d\times d}[\![Z-p, \bar{Z}-\bar{p}]\!]$ such that

$$\rho'\big(H(Z), \bar{H}(\bar{Z})\big) = a(Z, \bar{Z})\rho(Z, \bar{Z}).$$

If $(M, p)$ and $(M', p')$ are given in normal coordinates by $Q$ and $Q'$ respectively, we write $H(z, w) = (F(z, w), G(z, w))$, where $F$ is the first $n$ components and $G$ the last $d$ components of $H$. The condition that $H$ maps $M$ into $M'$ is then equivalent to that the following power-series identity is fulfilled:

$$Q'\big(F\big(z, Q(z, \chi, \tau)\big), \bar{H}(\chi, \tau)\big) = G\big(z, Q(z, \chi, \tau)\big). \tag{2.6}$$

In some of the results, we are going to assume that a mapping $H$ is generically of full rank. By this we mean that

$$\det \frac{\partial H}{\partial Z}(Z) \not\equiv 0. \tag{2.7}$$

We will use the following well-known result, which follows in a straightforward way from linear algebra.

**Lemma 2.1.** *With the above notation, if* (2.6) *is satisfied, then the condition*

$$\det H_Z(Z) \not\equiv 0 \tag{2.8}$$

*is equivalent to*

$$\det \frac{\partial}{\partial z} F\big(z, Q(z, \chi, \tau)\big) \not\equiv 0 \tag{2.9}$$

*and*

$$\det\left(\frac{\partial G}{\partial w}\big(z, Q(z, \chi, \tau)\big) - Q'_{z'}\big(F\big(z, Q(z, \chi, \tau)\big), \bar{H}(\chi, \tau)\big)\frac{\partial F}{\partial w}\big(z, Q(z, \chi, \tau)\big)\right) \not\equiv 0. \tag{2.10}$$

## 3. Finite jet determination results for systems of formal equations

We start by proving the following proposition, which we will be using twice in this paper:

**Proposition 3.1.** *Let $P(x, Y) = (P^1(x, Y), \ldots, P^N(x, Y))$ be a vector-valued power series in $(x, Y) \in \mathbb{C}^{n_1} \times \mathbb{C}^N$ for some integers $n_1 \geqslant 0$ and $N \geqslant 1$. Let $\varphi_0(x, t) = (\varphi_0^1(x, t), \ldots, \varphi_0^N(x, t))$ be a vector-valued power series in $(x, t) \in \mathbb{C}^{n_1} \times \mathbb{C}^d$ such that*

$$\det \frac{\partial P}{\partial Y}\big(x, \varphi_0(x, t)\big) \not\equiv 0. \tag{3.1}$$

*Let $\alpha^0$ be a multi-index in $n_1$ components and $\beta^0$ a multi-index with $d$ components such that*

$$\frac{\partial^{|\alpha^0|+|\beta^0|}}{\partial x^{\alpha^0}\partial t^{\beta^0}}\bigg|_{\substack{x=0\\t=0}} \det\frac{\partial P}{\partial Y}\big(x,\varphi_0(x,t)\big)\neq 0. \tag{3.2}$$

*Then for any power series $\varphi(x,t)$ satisfying*

(1) $\frac{\partial^{|\alpha|+|\beta|}\varphi}{\partial x^\alpha\partial t^\beta}(0)=\frac{\partial\varphi_0}{\partial x^\alpha\partial t^\beta}(0)$, *for* $|\alpha|\leqslant|\alpha^0|$, $|\beta|\leqslant|\beta^0|$,
(2) *for some $k$ we have* $\frac{\partial^{|\beta|}}{\partial t^\beta}|_{t=0}P(x,\varphi(x,t))=\frac{\partial^{|\beta|}}{\partial t^\beta}|_{t=0}P(x,\varphi_0(x,t))$ *for* $|\beta|\leqslant|\beta^0|+k$

*we have that*

$$\frac{\partial^{|\beta|}\varphi}{\partial t^\beta}(x,0)=\frac{\partial^{|\beta|}\varphi_0}{\partial t^\beta}(x,0),\quad \textit{for } |\beta|\leqslant k. \tag{3.3}$$

**Proof.** The proof is inspired by [20]. See also [19].

There exists a matrix-valued power series $A(x,Y,Z)$ such that

$$P(x,Y)-P(x,Z)=A(x,Y,Z)(Y-Z) \tag{3.4}$$

and

$$A(x,Y,Y)=\frac{\partial P}{\partial Y}(x,Y). \tag{3.5}$$

In (3.4), we treat $P$, $Y$ and $Z$ as column vectors.

We have

$$\begin{aligned}\frac{\partial P}{\partial Y}\big(x,\varphi_0(x,t)\big)&=A\big(x,\varphi_0(x,t),\varphi_0(x,t)\big)\\&=A\big(x,\varphi(x,t),\varphi_0(x,t)\big)+O\big(|x|^{|\alpha^0|+1},|t|^{|\beta^0|+1}\big).\end{aligned} \tag{3.6}$$

Therefore if we define

$$a(x,t)=\det A\big(x,\varphi(x,t),\varphi_0(x,t)\big), \tag{3.7}$$

we have

$$\frac{\partial^{|\alpha^0|+|\beta^0|}a}{\partial x^{\alpha^0}\partial t^{\beta^0}}(0)=\frac{\partial^{|\alpha^0|+|\beta^0|}}{\partial x^{\alpha^0}\partial t^{\beta^0}}\bigg|_{\substack{x=0\\t=0}}\det\frac{\partial P}{\partial Y}\big(x,\varphi_0(x,t)\big)\neq 0. \tag{3.8}$$

We get

$$\begin{aligned}O\big(|t|^{|\beta^0|+k+1}\big)&=P\big(x,\varphi(x,t)\big)-P\big(x,\varphi_0(x,t)\big)\\&=A\big(x,\varphi(x,t),\varphi_0(x,t)\big)\big(\varphi(x,t)-\varphi_0(x,t)\big).\end{aligned} \tag{3.9}$$

After multiplying both side with the classical adjoint of $A(x,\varphi(x,t),\varphi_0(x,t))$, we get

$$O\big(|t|^{|\beta^0|+k+1}\big)=a(x,t)\big(\varphi(x,t)-\varphi_0(x,t)\big). \tag{3.10}$$

From (3.8), we have that $\frac{\partial^{|\beta^0|}a}{\partial t^{\beta^0}}(x,0)\not\equiv 0$ and therefore

$$\frac{\partial^{|\beta|}\varphi}{\partial t^{\beta}}(x,0)\equiv\frac{\partial^{|\beta|}\varphi_0}{\partial t^{\beta}}(x,0)\quad\text{for } |\beta|\leqslant k. \qquad\Box \tag{3.11}$$

In Section 7, where we are working with hypersurfaces of infinite type, we need the following result on finite jet determination for systems of singular differential equations:

Let $K$ denote the field $\mathbb{R}$ or $\mathbb{C}$.

**Theorem 3.2.** *Consider a singular system of differential equations for a $K^n$-valued power series $y(x,s)$, where $s\in K$ and $x\in K^m$ of the form*

$$\partial_s y(x,s)=\frac{p(x,s,y(x,s))}{q(x,s,y(x,s))}, \tag{3.12}$$

*where $p(x,s,y)$ and $q(x,s,y)$ are power series (valued in $K^n$ and $K$, respectively). Let $y_0(x,s)$ with $y_0(0)=0$ be a power-series solution of* (3.12) *such that $q(x,s,y_0(x,s))\not\equiv 0$. Then there exists an integer $N\geqslant 0$ such that, if $y(x,s)$ is any solution of* (3.12) *with $\partial_s^k y(x,0)=\partial_s^k y_0(x,0)$ for $0\leqslant k\leqslant N$, then $y(x,s)=\hat{y}(x,s)$.*

The theorem can be reduced to the following special case. (We will include the details below.)

**Theorem 3.3** *(Ebenfelt, Lamel, Zaitsev). Consider a singular differential equation for a $K^n$-valued power series $y(x,s)$, where $s\in K$ and $x\in K^m$ of the form*

$$s^{\gamma+1}\partial_s y(x,s)=\frac{p(x,s,y(x,s))}{q(x,s,y(x,s))}, \tag{3.13}$$

*where $\gamma\geqslant 0$ is an integer, $p(x,s,y)$ and $q(x,s,y)$ are power series (valued in $K^n$ and $K$, respectively) with $q(x,0,0)\not\equiv 0$. Let $\hat{y}(x,s)$ be a power-series solution of* (3.13) *with $\hat{y}(x,0)\equiv 0$. Then there exists an integer $k\geqslant 0$ such that, if $y(x,s)$ is another solution of* (3.13) *with $\partial_s^l y(x,0)=\partial_s^l\hat{y}(x,0)$ for $0\leqslant l\leqslant k$, then $y(x,s)=\hat{y}(x,s)$.*

**Remark 3.4.** The theorem above, stated as Theorem 5.1 in [13], was in their paper formulated for real-analytic functions, but the proof works for power series as well. I also want to point out that the notation $y^{(k)}$ used in the proof of that theorem should have denoted the coefficients not derivatives. The theorem is also true for $\gamma=-1$. In this case, the system is nonsingular and the solution is unique.

**Proof of Theorem 3.2.** Let $l$ be the largest integer such that

$$\left.\frac{\partial^k}{\partial s^k}\right|_{s=0} q\big(x,s,y_0(x,s)\big)\equiv 0,\quad k<l. \tag{3.14}$$

Let

$$a(x,s)=\sum_{k=0}^{l}\frac{1}{k!}\frac{\partial^k y}{\partial s^k}(x,0)s^k. \tag{3.15}$$

We will write

$$y_0(x,s)=a(x,s)+s^l\tilde{y}_0(x,s), \tag{3.16}$$

so $\tilde{y}_0(x,0)\equiv 0$.

We will a priori assume that $N\geqslant l$, so we will only consider solutions $y(x,s)$ such that $\frac{\partial^k y}{\partial s^k}(x,0)=\frac{\partial^k y_0}{\partial s^k}(x,0)$ for $k\leqslant l$. That is, solutions that can be written as

$$y(x,s)=a(x,s)+s^l\tilde{y}(x,s), \tag{3.17}$$

where $\tilde{y}(x,0)\equiv 0$. By differentiating the above equation and using (3.13), we get

$$a_s(x,s)+ls^{l-1}\tilde{y}(x,s)+s^l\tilde{y}_s(x,s)=\frac{p(x,s,a(x,s)+s^l\tilde{y}(x,s))}{q(x,s,a(x,s)+s^l\tilde{y}(x,s))}. \tag{3.18}$$

Let the power series $Q(x,s,y)$ be defined by

$$Q(x,s,y)=q\big(x,s,a(x,s)+s^l y\big). \tag{3.19}$$

We see from the definition of the integer $l$ that $Q$ has the property

$$\frac{\partial^k Q}{\partial s^k}(x,0,y)\equiv 0,\quad k<l \tag{3.20}$$

and

$$\frac{\partial^l Q}{\partial s^l}(x,0,0)\not\equiv 0. \tag{3.21}$$

Therefore, we define the new power series $\tilde{Q}(x,s,y)$ by the equation

$$Q(x,s,y)=s^l\tilde{Q}(x,s,y). \tag{3.22}$$

Furthermore, we have that $\tilde{Q}(x,0,0)\not\equiv 0$.

From (3.18), we get that $\tilde{y}$ satisfies the following system of ODE's

$$s^{2l}\tilde{y}_s(x,s)=\frac{\tilde{P}(x,s,\tilde{y}(x,s))}{\tilde{Q}(x,s,\tilde{y}(x,s))}, \tag{3.23}$$

where the vector-valued power series $\tilde{P}(x,s,y)$ is defined by

$$\tilde{P}(x,s,y)=p\big(x,s,a(x,s)+s^l y\big)-s^l a_s(x,s)-ls^{2l-1}y. \tag{3.24}$$

Now, let $\tilde{N}$ be the integer we get by applying Theorem 3.3 to the system (3.23) with the distinguished solution $\tilde{y}_0(x,s)$. Let $N = \tilde{N} + l$. If $y(x,s)$ is a solution to the system (3.12) such that $\frac{\partial^k y}{\partial s^k}(x,0) = \frac{\partial^k y_0}{\partial s^k}(x,0)$ for $k \leqslant N$, and we define $\tilde{y}(x,s)$ by (3.17), then we have that $\frac{\partial^k \tilde{y}}{\partial s^k}(x,0) = \frac{\partial^k \tilde{y}_0}{\partial s^k}(x,0)$ for $k \leqslant \tilde{N}$. Thus, Theorem 3.3 gives us that $\tilde{y}(x,s) \equiv \tilde{y}_0(x,s)$, and hence that $y(x,s) \equiv y_0(x,s)$. $\square$

## 4. Universal polynomials and mapping identities

In this section, we will define some families of polynomials. These polynomials are used to establish mapping identities, i.e., power-series identities that are satisfied whenever we have a formal mapping from $M$ to $M'$, where $M$ and $M'$ are formal generic submanifolds in normal coordinates. The mapping identities will be used to prove partial jet determinations results in Sections 5 and 6. The polynomials do not depend on the manifolds or the mapping. Therefore, we call the polynomials universal.

We start by defining a polynomial for each tuple of indices. However, we will show in Lemma 4.1 and later in Lemma 4.3 that two tuples with the same components but in different order produce the same polynomial. Therefore, we will then switch to multi-index notation to denote the polynomials.

For any nonempty tuple of indices $A$ (i.e., $A = (j_1, \dots, j_l)$ for $l \geqslant 1$, where each $j_k \in \{1, 2, \dots, n\}$), we will define the universal polynomial $P_A((\Lambda_{\alpha'})_{1 \leqslant |\alpha'| \leqslant |A|})$ below, where $\Lambda_{\alpha'} \in \mathbb{C}^{n+1}$, $|A|$ denotes the number of components of the tuple $A$, and where $\alpha'$ is a multi-index (with $n$ components). Let $\Lambda_{\alpha'} = (\hat{\Lambda}_{\alpha'}, \Lambda_{\alpha'}^{n+1})$, where $\hat{\Lambda}_{\alpha'}$ denotes the first $n$ components of $\Lambda_{\alpha'}$. We will treat $\Lambda_{\alpha'}$ and $\hat{\Lambda}_{\alpha'}$ as column vectors with $n+1$ and $n$ components respectively. Let $e_k$ be the multi-index $(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in position $k$. By $(\hat{\Lambda}_{\alpha'})_{|\alpha'|=1}$ we will mean the $n \times n$ matrix $(\hat{\Lambda}_{e_1}, \dots, \hat{\Lambda}_{e_n})$. If $M$ is an $n \times n$ matrix, let $\mathrm{Adj}(M)$ denote the classical adjoint of $M$ and let $\mathrm{Adj}_j^k(M)$ be the $(k, j)$ entry of this matrix.

Let $q$ be the polynomial

$$q\big((\hat{\Lambda}_{\alpha'})_{|\alpha'|=1}\big) = \det\big((\hat{\Lambda}_{\alpha'})_{|\alpha'|=1}\big). \tag{4.1}$$

For tuples containing only one index, that is if $|A| = 1$ so $A = (j)$ for some $j \in \{1, \dots, n\}$, then the polynomials $P_A$ are defined by

$$P_{(j)}\big((\Lambda_{\alpha'})_{|\alpha'|=1}\big) = \sum_{k=1}^{n} \Lambda_{e_k}^{n+1} \, \mathrm{Adj}_j^k\big((\hat{\Lambda}_{\alpha'})_{|\alpha'|=1}\big). \tag{4.2}$$

For tuples of more than one component, the polynomials are defined recursively by the following formula. Here $(A, j)$ denotes the tuple we get by appending $j$ to the end of the tuple $A$:

$$\begin{aligned} P_{(A,j)}\big((\Lambda_{\alpha'})_{1 \leqslant |\alpha'| \leqslant |A|+1}\big) = \sum_{k=1}^{n} \Bigg( & q\big((\hat{\Lambda}_{\alpha'})_{|\alpha'|=1}\big) \sum_{1 \leqslant |\gamma| \leqslant |A|} \frac{\partial P_A}{\partial \Lambda_\gamma}\big((\Lambda_{\alpha'})_{1 \leqslant |\alpha'| \leqslant |A|}\big) \Lambda_{\gamma + e_k} \\ & - \big(2|A| - 1\big) P_A\big((\Lambda_{\alpha'})_{1 \leqslant |\alpha'| \leqslant |A|}\big) \sum_{|\gamma|=1} \frac{\partial q}{\partial \hat{\Lambda}_\gamma}\big((\hat{\Lambda}_{\alpha'})_{|\alpha'|=1}\big) \hat{\Lambda}_{\gamma + e_k} \Bigg) \\ & \times \mathrm{Adj}_j^k\big((\hat{\Lambda}_{\alpha'})_{|\alpha'|=1}\big). \end{aligned} \tag{4.3}$$

**Lemma 4.1.** *The polynomials $P_A$ are symmetric in the components of the tuple $A$. That is, if $A_1$ and $A_2$ are two tuples of indices with the same number of components and $A_2$ is a permutation of the entries in $A_1$, then $P_{A_1} = P_{A_2}$.*

**Proof.** We will suppress the arguments of the polynomials in order to make the notation a bit more compact. Also if the range of an index in a sum is not specified, we will assume that the sum is from 1 to $n$.

For $|A| \geqslant 0$, let

$$\tilde{P}_{(A,k_1,k_2)} = \sum_{j_1,j_2} P_{(A,j_1,j_2)} \Lambda^{j_1}_{e_{k_1}} \Lambda^{j_2}_{e_{k_2}}. \tag{4.4}$$

We see that

$$P_{(A,j_1,j_2)} q^2 = \sum_{k_1,k_2} \tilde{P}_{(A,k_1,k_2)} \operatorname{Adj}^{k_1}_{j_1} \operatorname{Adj}^{k_2}_{j_2}, \tag{4.5}$$

so $P_{(A,j_1,j_2)}$ is symmetric in $j_1$, $j_2$ if and only if $\tilde{P}_{(A,j_1,j_2)}$ is symmetric in $j_1$, $j_2$.

We will need the following identity:

$$\delta^{k_1}_{p_1} \sum_{s_1,t_1} \frac{\partial q}{\partial \Lambda^{s_1}_{e_{t_1}}} \Lambda^{s_1}_{e_{t_1}+e_{p_2}} = \sum_{s_1,t_1,j_1} \frac{\partial \operatorname{Adj}^{k_1}_{j_1}}{\partial \Lambda^{s_1}_{e_{t_1}}} \Lambda^{j_1}_{e_{p_1}} \Lambda^{s_1}_{e_{t_1}+e_{p_2}} + \sum_{j_1} \operatorname{Adj}^{k_1}_{j_1} \Lambda^{j_1}_{e_{p_1}+e_{p_2}}. \tag{4.6}$$

This follows by applying the vector field $\sum_{s_1,t_1} \Lambda^{s_1}_{e_{t_1}+e_{p_2}} \frac{\partial}{\partial \Lambda^{s_1}_{e_{t_1}}}$ to the identity

$$\delta^{k_1}_{p_1} q = \sum_{j_1} \operatorname{Adj}^{k_1}_{j_1} \Lambda^{j_1}_{e_{p_1}}. \tag{4.7}$$

We are going to start by proving that $\tilde{P}_{(p_1,p_2)}$ is symmetric in $p_1$ and $p_2$.

From (4.2) and (4.4), we get

$$\tilde{P}_{(p_1,p_2)} = q^3 \Lambda^{n+1}_{e_{p_1}+e_{p_2}} + q^2 \sum_{k_1,j_1,s_1,t_1} \Lambda^{n+1}_{e_{k_1}} \frac{\partial \operatorname{Adj}^{k_1}_{j_1}}{\partial \Lambda^{s_1}_{e_{t_1}}} \Lambda^{s_1}_{e_{t_1}+e_{p_2}} \Lambda^{j_1}_{e_{p_1}} - q^2 \Lambda^{n+1}_{e_{p_1}} \sum_{s_1,t_1} \frac{\partial q}{\partial \Lambda^{s_1}_{e_{t_1}}} \Lambda^{s_1}_{e_{p_2}}. \tag{4.8}$$

Now using (4.6), we get

$$\tilde{P}_{(p_1,p_2)} = q^3 \Lambda^{n+1}_{e_{p_1}+e_{p_2}} - q^2 \sum_{k_1,j_1} \Lambda^{n+1}_{e_{k_1}} \operatorname{Adj}^{k_1}_{j_1} \Lambda^{j_1}_{e_{p_1}+e_{p_2}}. \tag{4.9}$$

From this equation, we see that $\tilde{P}_{(p_1,p_2)}$ is symmetric in $p_1$ and $p_2$.

Now assume the length of $A$ is greater than 0. From (4.3), we get that

$$P_{(A,j_1)} = \sum_{k_1} Q_{A,k_1} \operatorname{Adj}^{k_1}_{j_1}, \tag{4.10}$$

where

$$Q_{A,p_1} = q \sum_{1\leqslant|\gamma|\leqslant|A|} \frac{\partial P_A}{\partial \Lambda_\gamma} \Lambda_{\gamma+e_{p_1}} - \big(2|A|-1\big) P_A \sum_{s_1,t_1} \frac{\partial q}{\partial \Lambda^{s_1}_{t_1}} \Lambda^{s_1}_{e_{t_1}+e_{p_1}}. \tag{4.11}$$

Using (4.3) once again on (4.10) and using (4.4), we get

$$\tilde{P}_{(A,p_1,p_2)} = q^3 \sum_{1\leqslant|\gamma|\leqslant|A|+1} \frac{\partial Q_{A,p_1}}{\partial \Lambda_\gamma} \Lambda_{\gamma+e_{p_2}} + q^2 \sum_{k_1,j_1,s_1,t_1} Q_{A,k_1} \frac{\partial \operatorname{Adj}^{k_1}_{j_1}}{\partial \Lambda^{s_1}_{e_{t_1}}} \Lambda^{j_1}_{e_{p_1}} \Lambda^{s_1}_{e_{t_1}+e_{p_2}}$$
$$- \big(2|A|+1\big) q^2 Q_{A,p_1} \sum_{s_1,t_1} \frac{\partial q}{\partial \Lambda^{s_1}_{t_1}} \Lambda^{s_1}_{e_{t_1}+e_{p_2}}.$$

Now using (4.6) on the above equation, we get

$$\tilde{P}_{(A,p_1,p_2)} = q^3 \sum_{1\leqslant|\gamma|\leqslant|A|+1} \frac{\partial Q_{A,p_1}}{\partial \Lambda_\gamma} \Lambda_{\gamma+e_{p_2}} - 2|A| q^2 Q_{A,p_1} \sum_{s_1,t_1} \frac{\partial q}{\partial \Lambda^{s_1}_{t_1}} \Lambda^{s_1}_{e_{t_1}+e_{p_2}}$$
$$- q^2 \sum_{k_1,j_1} Q_{A,k_1} \operatorname{Adj}^{k_1}_{j_1} \Lambda^{j_1}_{e_{p_1}+e_{p_2}}. \tag{4.12}$$

After using (4.11) and collecting terms, we finally get

$$\begin{aligned}
\tilde{P}_{(A,p_1,p_2)} = {} & -q^3\big(2|A|-1\big) \sum_{1\leqslant|\gamma|\leqslant|A|} \frac{\partial P_A}{\partial \Lambda_{\gamma+e_{p_1}}} \sum_{s_1,t_1} \frac{\partial q}{\partial \Lambda^{s_1}_{e_{t_1}}} \Lambda^{s_1}_{e_{t_1}+e_{p_2}} \\
& - q^3\big(2|A|-1\big) \sum_{1\leqslant|\gamma|\leqslant|A|} \frac{\partial P_A}{\partial \Lambda_{\gamma+e_{p_2}}} \sum_{s_1,t_1} \frac{\partial q}{\partial \Lambda^{s_1}_{e_{t_1}}} \Lambda^{s_1}_{e_{t_1}+e_{p_1}} \\
& + q^4 \sum_{1\leqslant|\gamma^1|,|\gamma^2|\leqslant|A|} \sum_{1\leqslant s_1,s_2\leqslant n+1} \frac{\partial^2 P_A}{\partial \Lambda^{s_1}_{\gamma^1} \partial \Lambda^{s_2}_{\gamma^2}} \Lambda^{s_1}_{\gamma^1+e_{p_1}} \Lambda^{s_2}_{\gamma^2+e_{p_2}} \\
& + q^4 \sum_{1\leqslant|\gamma|\leqslant|A|} \frac{\partial P_A}{\partial \Lambda_\gamma} \Lambda_{\gamma+e_{p_1}+e_{p_2}} \\
& - \big(2|A|-1\big) q^3 P_A \sum_{s_1,s_2,t_1,t_2} \frac{\partial^2 q}{\partial \Lambda^{s_1}_{t_1} \partial \lambda^{s_2}_{t_2}} \Lambda^{s_1}_{e_{t_1}+e_{p_1}} \Lambda^{s_2}_{e_{t_2}+e_{p_2}} \\
& - \big(2|A|-1\big) q^3 P_A \sum_{s_1,t_1} \frac{\partial q}{\partial \Lambda^{s_1}_{e_{t_1}}} \Lambda^{s_1}_{e_{t_1}+e_{p_1}+e_{p_2}} \\
& - 2|A|\big(2|A|-1\big) q^2 P_A \sum_{s_1,s_2,t_1,t_2} \frac{\partial q}{\partial^{s_1}_{e_{t_1}}} \Lambda^{s_1}_{e_{t_1}+e_{p_1}} \frac{\partial q}{\partial^{s_2}_{e_{t_2}}} \Lambda^{s_2}_{e_{t_2}+e_{p_2}} \\
& - q^2 \sum_{k_1,j_1} Q_{A,k_1} \operatorname{Adj}^{k_1}_{j_1} \Lambda^{j_1}_{e_{p_1}+e_{p_2}}.
\end{aligned} \tag{4.13}$$

From this expression, it is clear that $\tilde{P}_{(A,p_1,p_2)} = \tilde{P}_{(A,p_2,p_1)}$, so $P_{(A,p_1,p_2)} = P_{(A,p_2,p_1)}$ for any tuple $A$. By induction over $|A|$ using the recursion formula (4.3), we see that $P_A$ is symmetric in all its indices. $\square$

Because of the symmetry, we will switch to multi-index notation to denote the polynomials. That is, $P_\alpha((\Lambda_{\alpha'})_{1\leqslant|\alpha'|\leqslant|\alpha|}) = P_A((\Lambda_{\alpha'})_{1\leqslant|\alpha'|\leqslant|A|})$ for a tuple $A$ containing exactly $\alpha_j$ copies of the integer $j$ for $1 \leqslant j \leqslant n$.

The usefulness of these polynomials is given by the following lemma:

**Lemma 4.2.**

(a) *Let $H = (F, G)$ be a formal mapping between two formal generic manifolds $M$ and $M'$ in $\mathbb{C}^{n+d}$ of real codimension $d$ given in normal coordinates by $Q$ and $Q'$ respectively (i.e., the mapping condition* (2.6) *is satisfied). Assume that the mapping $H$ satisfies the condition* $\det \frac{\partial}{\partial z} F(z, Q(z,\chi,\tau)) \not\equiv 0$. *Then for $1 \leqslant r \leqslant d$ and for every multi-index $\alpha$ with $|\alpha| > 0$, we have the following identity in the quotient field of formal power series:*

$$Q'^r_{z'^\alpha}\big(F\big(z, Q(z,\chi,\tau)\big), \bar{H}(\chi,\tau)\big) = \frac{P_\alpha\big(\big(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}} H^{(r)}(z, Q(z,\chi,\tau))\big)_{1\leqslant|\alpha'|\leqslant|\alpha|}\big)}{(\det \frac{\partial}{\partial z} F(z, Q(z,\chi,\tau)))^{2|\alpha|-1}}, \tag{4.14}$$

*where $H^{(r)}(z,w) = (F(z,w), G^r(z,w))$, where $G^r$ is the $r$th component of $G$, and where $Q'^r$ is the $r$th component of $Q'$.*

(b) *If, furthermore, $q_0 \geqslant 0$ is the largest integer such that*

$$\frac{\partial^{|\eta'|+|\delta'|}}{\partial z^{\eta'}\partial\tau^{\delta'}}\bigg|_{\substack{z=0\\ \tau=0}} \det \frac{\partial}{\partial z} F\big(z, Q(z,\chi,\tau)\big) \equiv 0$$

*for all $\eta'$ and $\delta'$ with $|\eta'|+|\delta'| < q_0$, then $\frac{\partial^{|\eta|+|\delta|}}{\partial z^{\eta}\partial\tau^{\delta}}\big|_{\substack{z=0\\ \tau=0}} P_\alpha\big(\big(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}} H^{(r)}(z, Q(z,\chi,\tau))\big)_{1\leqslant|\alpha'|\leqslant|\alpha|}\big)$ is a polynomial in $\frac{\partial^{|\eta'|+|\delta'|}}{\partial z^{\eta'}\partial\tau^{\delta'}}\big|_{\substack{z=0\\ \tau=0}} \frac{\partial}{\partial z} H^{(r)}(z, Q(z,\chi,\tau))$ for $|\eta'| + |\delta'| \leqslant |\eta| + |\delta| - (|\alpha| - 1)(q_0 - 1)$.*

**Proof.** Let the power series $\lambda(z,\chi,\tau)$ be defined by

$$\lambda(z,\chi,\tau) = \det \frac{\partial}{\partial z} F\big(z, Q(z,\chi,\tau)\big). \tag{4.15}$$

Now, let us differentiate the $r$th component of (2.6) with respect to $z$ and write the result as a row-vector:

$$Q'^r_{z'}\big(F\big(z, Q(z,\chi,\tau)\big), \bar{H}(\chi,\tau)\big) \frac{\partial}{\partial z} F\big(z, Q(z,\chi,\tau)\big) = \frac{\partial}{\partial z} G^r\big(z, Q(z,\chi,\tau)\big). \tag{4.16}$$

Note that we consider $F(z, Q(z,\chi,\tau))$ as a column vector, so $\frac{\partial}{\partial z} F(z, Q(z,\chi,\tau))$ is an $n \times n$ matrix and $\frac{\partial}{\partial z} G^r(z, Q(z,\chi,\tau))$ is a row vector.

After multiplying both sides with the classical adjoint of $\frac{\partial}{\partial z}F(z, Q(z,\chi,\tau))$ and dividing with the determinant $\lambda(z,\chi,\tau)$, we get

$$Q'^{r}_{z'}\big(F\big(z, Q(z,\chi,\tau)\big), \bar{H}(\chi,\tau)\big) = \frac{\frac{\partial}{\partial z}G^r(z, Q(z,\chi,\tau)) \cdot \operatorname{Adj} \frac{\partial}{\partial z}F(z, Q(z,\chi,\tau))}{\lambda(z,\chi,\tau)}. \tag{4.17}$$

Taking out the $k$th column of the above equation, we get

$$Q'^{r}_{z'_k}\big(F\big(z, Q(z,\chi,\tau)\big), \bar{H}(\chi,\tau)\big) = \frac{\frac{\partial}{\partial z}G^r(z, Q(z,\chi,\tau)) \cdot \operatorname{Adj}_k \frac{\partial}{\partial z}F(z, Q(z,\chi,\tau))}{\lambda(z,\chi,\tau)}, \tag{4.18}$$

where $\operatorname{Adj}_k$ denotes the $k$th column of the classical adjoint matrix. From (4.2), we see that the lemma holds for $\alpha = e_k$.

For notational convenience, let

$$\hat{P}_\alpha(z,\chi,\tau) = P_\alpha\left(\left(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}}H^{(r)}\big(z, Q(z,\chi,\tau)\big)\right)_{1\leqslant|\alpha'|\leqslant|\alpha|}\right).$$

Assume, for sake of induction, that the lemma holds for all $\alpha$ with $|\alpha| = k_0$ for some integer $k_0 \geqslant 1$. We will show that the lemma holds for $\alpha + e_q$.

By differentiating both sides of (4.14), we get

$$\begin{aligned}
&\frac{\partial Q'^{r}_{z'^\alpha}}{\partial z'}\big(F\big(z, Q(z,\chi,\tau)\big), \bar{H}(\chi,\tau)\big)\frac{\partial}{\partial z}F\big(z, Q(z,\chi,\tau)\big) \\
&\quad = \frac{\lambda(z,\chi,\tau)\frac{\partial \hat{P}_\alpha}{\partial z}(z,\chi,\tau) - (2|\alpha|-1)\hat{P}_\alpha(z,\chi,\tau)\frac{\partial\lambda}{\partial z}(z,\chi,\tau)}{\lambda(z,\chi,\tau)^{2|\alpha|}}.
\end{aligned} \tag{4.19}$$

By multiplying with the adjoint matrix and dividing with the determinant $\lambda(z,\chi,\tau)$, we see after looking at the different components of the resulting equation that for $1 \leqslant q \leqslant n$, we have

$$Q'^{r}_{z'^{\alpha+e_q}}\big(F\big(z, Q(z,\chi,\tau)\big), \bar{H}(\chi,\tau)\big) = \frac{A(z,\chi,\tau)}{\lambda(z,\chi,\tau)^{2(|\alpha|+1)-1}}, \tag{4.20}$$

where

$$\begin{aligned}
A(z,\chi,\tau) = {}&\left(\lambda(z,\chi,\tau)\frac{\partial \hat{P}_\alpha}{\partial z}(z,\chi,\tau) - \big(2|\alpha|-1\big)\hat{P}_\alpha(z,\chi,\tau)\frac{\partial\lambda}{\partial z}(z,\chi,\tau)\right) \\
&\times \operatorname{Adj}_q \frac{\partial}{\partial z}F\big(z, Q(z,\chi,\tau)\big).
\end{aligned} \tag{4.21}$$

By comparing (4.21) with (4.3), we see that $A(z,\chi,\tau) = \hat{P}_{\alpha+e_q}(z,\chi,\tau)$.

Because $Q'^{r}_{z'^{\alpha+e_q}}(F(z,Q(z,\chi,\tau)),\bar{H}(\chi,\tau))$ is a power series, we have that $\frac{\partial^{|\eta|+|\delta|}\hat{P}_\alpha}{\partial z^\eta \partial\tau^\delta}(0,\chi,0)\equiv 0$ for $|\eta|+|\delta|<(2|\alpha|-1)q_0$. From (4.21), we now see that $\frac{\partial^{|\eta|+|\delta|}\hat{P}_{\alpha+e_q}}{\partial z^\eta \partial\tau^\delta}(0,\chi,0)$ is a polynomial in $\frac{\partial^{|\eta^2|+|\delta^2|}\hat{P}_\alpha}{\partial z^{\eta^2}\partial\tau^{\delta^2}}(0,\chi,0)$ and $\frac{\partial^{|\eta^3|+|\delta^3|}\lambda}{\partial z^{\eta^3}\partial\tau^{\delta^3}}(0,\chi,0)$ for $|\eta^2|+|\delta^2|\leqslant|\eta|+|\delta|-q_0+1$ and $|\eta^3|+|\delta^3|\leqslant|\eta|+|\delta|-(2|\alpha|-1)q_0+1$. Now by using the induction hypothesis, we see that $\frac{\partial^{|\eta|+|\delta|}\hat{P}_{\alpha+e_q}}{\partial z^\eta \partial\tau^\delta}(0,\chi,0)$ is a polynomial in $\frac{\partial^{|\eta'|+|\delta'|}}{\partial z^{\eta'}\partial\tau^{\delta'}}\big|_{\substack{z=0\\ \tau=0}}\frac{\partial}{\partial z}H(z,Q(z,\chi,\tau))$ for $|\eta'|+|\delta'|\leqslant|\eta|+|\delta|-q_0+1-(|\alpha|-1)(q_0-1)=|\eta|+|\delta|-|\alpha|(q_0-1)$. We have proved that the lemma holds for $\alpha+e_q$. By taking different $\alpha$ with $|\alpha|=k_0$ and different $q$, we see that the lemma holds for all $\alpha$ with $|\alpha|=k_0+1$. □

We are now going to introduce another set of universal polynomials. For every nonzero multi-index $\alpha$ with $n$ components and tuple $B$ of indices ($|B|\geqslant 0$), where each index ranges over the set $\{1,\dots,d\}$, we are going to define the polynomial $P_{\alpha,B}((\Lambda_{\alpha',\beta'})_{\substack{1\leqslant|\alpha'|\leqslant|\alpha|\\ |\beta'|\leqslant|B|}})$, where the variables $\Lambda_{\alpha',\beta'}\in\mathbb{C}^{n+1}$. In expressions like the one above, $1\leqslant|\alpha'|\leqslant|\alpha|$ and $|\beta'|\leqslant|B|$ mean that $\alpha'$ is a multi-index with $n$ components, $\beta'$ is a multi-index with $d$ components and that they run over their indicated ranges.

To make the expressions a bit more compact, let for any nonzero multi-index $\alpha$ (with $n$ components)

$$k_\alpha=2|\alpha|-1. \tag{4.22}$$

For the empty tuple $B=\emptyset$ (i.e., when $|B|=0$), the polynomials $P_{\alpha,B}$ are defined by

$$P_{\alpha,\emptyset}\big((\Lambda_{\alpha',0})_{1\leqslant|\alpha'|\leqslant|\alpha|}\big)=P_\alpha\big((\Lambda_{\alpha',0})_{1\leqslant|\alpha'|\leqslant|\alpha|}\big). \tag{4.23}$$

For nonempty tuples, the polynomials are defined by the recursion formula below. We use the same notation as in the definition of $P_\alpha$. The polynomial $q$ is defined by (4.1). Let $|B|\geqslant 0$, then the polynomials $P_{\alpha,(B,j)}$ are defined recursively by

$$\begin{aligned}
&P_{\alpha,(B,j)}\big((\Lambda_{\alpha',\beta'})_{\substack{1\leqslant|\alpha'|\leqslant|\alpha|\\ |\beta'|\leqslant|B|+1}}\big)\\
&\quad=q\big((\hat{\Lambda}_{\alpha',0})_{|\alpha'|=1}\big)\sum_{\substack{1\leqslant|\gamma|\leqslant|\alpha|\\ |\delta|\leqslant|\beta|}}\frac{\partial P_{\alpha,B}}{\partial\Lambda_{\gamma,\delta}}\big((\Lambda_{\alpha',\beta'})_{\substack{1\leqslant|\alpha'|\leqslant|\alpha|\\ |\beta'|\leqslant|B|}}\big)\Lambda_{\gamma,\delta+e_j}\\
&\qquad-\big(k_\alpha+|B|\big)P_{\alpha,B}\big((\Lambda_{\alpha',\beta'})_{\substack{1\leqslant|\alpha'|\leqslant|\alpha|\\ |\beta'|\leqslant|B|}}\big)\sum_{|\gamma|=1}\frac{\partial q}{\partial\hat{\Lambda}_{\gamma,0}}\big((\hat{\Lambda}_{\alpha',0})_{|\alpha'|=1}\big)\hat{\Lambda}_{\gamma,e_j}.
\end{aligned}\tag{4.24}$$

**Lemma 4.3.** *The polynomials $P_{\alpha,B}$ are symmetric in the indices in the tuple $B$.*

**Proof.** As in the proof of Lemma 4.1, we will suppress the arguments to make the notation more compact.

Let $|B| \geqslant 0$. From the definition, we have

$$P_{\alpha,(B,p_1)} = q \sum_{\substack{1\leqslant|\gamma|\leqslant|\alpha| \\ |\delta|\leqslant|B|}} \frac{\partial P_{\alpha,B}}{\partial \Lambda_{\gamma,\delta}} \Lambda_{\gamma,\delta+e_{p_1}} - \big(k_\alpha + |B|\big) P_{\alpha,B} \sum_{|\gamma|=1} \frac{\partial q}{\partial \hat{\Lambda}_{\gamma,0}} \hat{\Lambda}_{\gamma,e_{p_1}}. \tag{4.25}$$

After using the recursion formula one more time and collecting terms, we get

$$\begin{aligned}
P_{\alpha,(B,p_1,p_2)} = q^2 &\sum_{\substack{1\leqslant|\gamma^1|,|\gamma^2|\leqslant|\alpha| \\ |\delta^1|,|\delta^2|\leqslant|B| \\ 1\leqslant k_1,k_2\leqslant n+1}} \frac{\partial^2 P_{\alpha,B}}{\partial \Lambda^{k_1}_{\gamma^1,\delta^1} \partial \Lambda^{k_2}_{\gamma^2,\delta^2}} \Lambda^{k_1}_{\gamma^1,\delta^1+e_{p_1}} \Lambda^{k_2}_{\gamma^2,\delta^2+e_{p_2}} \\
&+ q^2 \sum_{\substack{1\leqslant|\gamma|\leqslant|\alpha| \\ |\delta|\leqslant|B|}} \frac{\partial P_{\alpha,B}}{\partial \Lambda_{\gamma,\delta}} \Lambda_{\gamma,\delta+e_{p_1}+e_{p_2}} \\
&- \big(k_\alpha + |B|\big) q \sum_{\substack{1\leqslant|\gamma|\leqslant|\alpha| \\ |\delta|\leqslant|B|}} \frac{\partial P_{\alpha,B}}{\partial \Lambda_{\gamma,\delta}} \Lambda_{\gamma,\delta+e_{p_1}} \sum_{|\gamma|=1} \frac{\partial q}{\partial \hat{\Lambda}_{\gamma,0}} \hat{\Lambda}_{\gamma,e_{p_2}} \\
&- \big(k_\alpha + |B|\big) q \sum_{\substack{1\leqslant|\gamma|\leqslant|\alpha| \\ |\delta|\leqslant|B|}} \frac{\partial P_{\alpha,B}}{\partial \Lambda_{\gamma,\delta}} \Lambda_{\gamma,\delta+e_{p_2}} \sum_{|\gamma|=1} \frac{\partial q}{\partial \hat{\Lambda}_{\gamma,0}} \hat{\Lambda}_{\gamma,e_{p_1}} \\
&- \big(k_\alpha + |B|\big) q P_{\alpha,B} \sum_{\substack{|\gamma^1|,|\gamma^2|=1 \\ 1\leqslant s_1,s_2\leqslant n}} \frac{\partial^2 q}{\partial \Lambda^{s_1}_{\gamma^1,0} \partial \Lambda^{s_2}_{\gamma^2,0}} \Lambda^{s_1}_{\gamma^1,e_{p_1}} \Lambda^{s_2}_{\gamma^2,e_{p_2}} \\
&- \big(k_\alpha + |B|\big) q P_{\alpha,B} \sum_{|\gamma|=1} \frac{\partial q}{\partial \hat{\Lambda}_{\gamma,0}} \hat{\Lambda}_{\gamma,e_{p_1}+e_{p_2}} \\
&- \big(k_\alpha + |B| + 1\big)\big(k_\alpha + |B|\big) P_{\alpha,B} \sum_{|\gamma^1|,|\gamma^2|=1} \frac{\partial q}{\partial \hat{\Lambda}_{\gamma^1,0}} \hat{\Lambda}_{\gamma^1,e_{p_1}} \frac{\partial q}{\partial \hat{\Lambda}_{\gamma^2,0}} \hat{\Lambda}_{\gamma^2,e_{p_2}}.
\end{aligned} \tag{4.26}$$

From this equation, we see that $P_{\alpha,(B,p_1,p_2)} = P_{\alpha,(B,p_2,p_1)}$. We have proved that for any tuple $B$ that $P_{\alpha,B}$ is symmetric in the last two indices of $B$. By induction over $|B|$ using the recursion formula (4.24), it follows that $P_{\alpha,B}$ is symmetric in all the indices of $B$. $\square$

From the symmetry, it is motivated to use multi-index notation, so for any nonzero multi-index $\alpha$ with $n$ components and multi-index $\beta$ with $d$ components, we let $P_{\alpha,\beta}((\Lambda_{\alpha',\beta'})_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|}}) = P_{\alpha,B}((\Lambda_{\alpha',\beta'})_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|B|}})$, where $B$ is a tuple that has exactly $\beta_j$ copies of the integer $j$ for $1 \leqslant j \leqslant d$.

The polynomials $P_{\alpha,\beta}((\Lambda_{\alpha',\beta'})_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|}})$ have the following fundamental property:

**Lemma 4.4.**

(a) *Let $h_\alpha(x,t) = (\hat{h}_\alpha(x,t), h_\alpha^{n+1}(x,t))$ for $|\alpha| \geqslant 1$ be a family of vector-valued power series (indexed over the set of nonzero multi-indices with $n$ components), where the variables $x$ and $t$ have $n_1 \geqslant 0$ and $d \geqslant 1$ components respectively. Assume that $q((\hat{h}_\alpha(x,t))_{|\alpha|=1}) \not\equiv 0$. Then for any $\alpha$, $\beta$ with $|\alpha| \geqslant 1$ and $|\beta| \geqslant 0$, the following identity holds in the quotient field of formal power series:*

$$\frac{\partial^{|\beta|}}{\partial t^\beta}\left(\frac{P_{\alpha,0}((h_{\alpha'}(x,t))_{1\leqslant|\alpha'|\leqslant|\alpha|})}{q((\hat{h}_{\alpha'}(x,t))_{|\alpha'|=1})^{k_\alpha}}\right) = \frac{P_{\alpha,\beta}((\frac{\partial^{|\beta'|}h_{\alpha'}}{\partial t^{\beta'}}(x,t))_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|}})}{q((\hat{h}_{\alpha'}(x,t))_{|\alpha'|=1})^{k_\alpha+|\beta|}}. \tag{4.27}$$

(b) *Assume that the quotient $\frac{P_{\alpha,0}((h_{\alpha'}(x,t))_{1\leqslant|\alpha'|\leqslant|\alpha|})}{q((\hat{h}_{\alpha'}(x,t))_{|\alpha'|=1})^{k_\alpha}}$ is a power series (i.e., the denominator divides the numerator in the ring of formal power series). Let $x = (\hat{x}, y)$, where $\hat{x}$ has $n_2$ components and $y$ has $n_3$ components, where $n_2 + n_3 = n_1$ ($n_2$ or $n_3$ might be 0), and let $\tilde{q}$ be the largest integer such that $\frac{\partial^{|\eta'|+|\delta'|}}{\partial y^{\eta'}\partial t^{\delta'}}|_{\substack{y=0\\t=0}} q((\hat{h}_{\alpha'}(\hat{x},y,t))_{|\alpha'|=1}) \equiv 0$ for all $\eta'$ and $\delta'$ with $|\eta'| + |\delta'| < \tilde{q}$. Then for each $\eta$ and $\delta$, we have that $\frac{\partial^{|\eta|+|\delta|}}{\partial y^{\eta}\partial t^{\delta}}|_{\substack{y=0\\t=0}} P_{\alpha,\beta}((\frac{\partial^{|\beta'|}h_{\alpha'}}{\partial t^{\beta'}}(x,t))_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|}})$ is a polynomial in $\frac{\partial^{|\eta^1|+|\delta^1|}}{\partial y^{\eta^1}\partial t^{\delta^1}}|_{\substack{y=0\\t=0}} P_{\alpha,0}((h_{\alpha'}(x,t))_{1\leqslant|\alpha'|\leqslant|\alpha|})$ and $\frac{\partial^{|\eta^2|+|\delta^2|}}{\partial y^{\eta^2}\partial t^{\delta^2}}|_{\substack{y=0\\t=0}} q((\hat{h}_{\alpha'}(\hat{x},y,t))_{|\alpha'|=1})$ for $|\eta^1| + |\delta^1| \leqslant |\eta| + |\delta| - |\beta|(\tilde{q}-1)$ and $|\eta^2| + |\delta^2| \leqslant |\eta| + |\delta| - (k_\alpha - 1)\tilde{q} - |\beta|(\tilde{q}-1)$.*

**Proof.** From the recursion formula (4.24), we see that

$$\begin{aligned} &P_{\alpha,\beta+e_j}\left(\left(\frac{\partial^{|\beta'|}h_{\alpha'}}{\partial t^{\beta'}}(x,t)\right)_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|+1}}\right) \\ &\quad= q\big(\big(\hat{h}_{\alpha'}(x,t)\big)_{|\alpha'|=1}\big)\frac{\partial}{\partial t_j} P_{\alpha,\beta}\left(\left(\frac{\partial^{|\beta'|}h_{\alpha'}}{\partial t^{\beta'}}(x,t)\right)_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|}}\right) \\ &\qquad- \big(k_\alpha + |\beta|\big)P_{\alpha,\beta}\left(\left(\frac{\partial^{|\beta'|}h_{\alpha'}}{\partial t^{\beta'}}(x,t)\right)_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|}}\right)\frac{\partial}{\partial t_j} q\big(\big(\hat{h}_{\alpha'}(x,t)\big)_{|\alpha'|=1}\big), \end{aligned} \tag{4.28}$$

so we have that

$$\frac{\partial}{\partial t_j}\left(\frac{P_{\alpha,\beta}((\frac{\partial^{|\beta'|}h_{\alpha'}}{\partial t^{\beta'}}(x,t))_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|}})}{q((\hat{h}_{\alpha'}(x,t))_{|\alpha'|=1})^{k_\alpha+|\beta|}}\right) = \frac{P_{\alpha,\beta+e_j}((\frac{\partial^{|\beta'|}h_{\alpha'}}{\partial t^{\beta'}}(x,t))_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|+1}})}{q((\hat{h}_{\alpha'}(x,t))_{|\alpha'|=1})^{k_\alpha+|\beta|+1}}. \tag{4.29}$$

The first part of the lemma now follows by induction over the order of $|\beta|$.

To prove the second part of the lemma, we first note that the assumption gives us that the left hand side of (4.27) is a power series, so the right hand side is also a power series. Thus,

$\frac{\partial^{|\eta'|+|\beta'|}}{\partial y^{\eta'}\partial t^{\beta'}}\Big|_{\substack{y=0\\t=0}} P_{\alpha,\beta}((\frac{\partial^{|\beta'|}h_{\alpha'}}{\partial t^{\beta'}}(x,t))_{\substack{1\leqslant|\alpha'|\leqslant|\alpha|\\|\beta'|\leqslant|\beta|}}) \equiv 0$ for all $\eta'$ and $\beta'$ with $|\eta'|+|\beta'| < \tilde{q}(k_\alpha+|\beta|)$. The conclusion follows by induction over $|\beta|$ using the identity (4.28). □

We use these polynomials to get another set of mapping identities:

**Lemma 4.5.** *Let $H=(F,G)$ be a formal mapping between two formal generic manifolds $M$ and $M'$ in $\mathbb{C}^{n+d}$ of real codimension $d$ given in normal coordinates by $Q$ and $Q'$ respectively. That is, the mapping condition* (2.6) *is satisfied. Assume that the mapping $H$ satisfies the condition* $\det\frac{\partial}{\partial z}F(z,Q(z,\chi,\tau))\not\equiv 0$. *Then for every $1\leqslant r\leqslant d$ and nonzero multi-index $\alpha$ of $n$ components and multi-index $\beta$ of $d$ components* ($|\beta|\geqslant 0$)*, we have the following identity in the quotient field of power series:*

$$\frac{\partial^{|\beta|}}{\partial\tau^\beta} Q'^{r}_{z'^\alpha}\big(F\big(z,Q(z,\chi,\tau)\big),\bar{H}(\chi,\tau)\big) = \frac{P_{\alpha,\beta}((\frac{\partial^{|\alpha'|+|\beta'|}}{\partial z^{\alpha'}\partial\tau^{\beta'}}H^{(r)}(z,Q(z,\chi,\tau)))_{\substack{1\leqslant|\alpha'|\leqslant|\alpha|\\|\beta'|\leqslant|\beta|}})}{(\det\frac{\partial}{\partial z}F(z,Q(z,\chi,\tau)))^{2|\alpha|+|\beta|-1}}, \tag{4.30}$$

*where as in Lemma* 4.2*, $H^{(r)}(z,w)=(F(z,w),G^r(z,w))$. Furthermore,*

$$\frac{\partial^{|\eta|+|\delta|}}{\partial z^\eta\partial\tau^\delta}\bigg|_{\substack{z=0\\\tau=0}} P_{\alpha,\beta}\bigg(\bigg(\frac{\partial^{|\alpha'|+|\beta'|}}{\partial z^{\alpha'}\partial\tau^{\beta'}}H^{(r)}\big(z,Q(z,\chi,\tau)\big)\bigg)_{\substack{1\leqslant|\alpha'|\leqslant|\alpha|\\|\beta'|\leqslant|\beta|}}\bigg)$$

*is a polynomial in*

$$\frac{\partial^{|\eta'|+|\delta'|}}{\partial z^{\eta'}\partial\tau^{\delta'}}\bigg|_{\substack{z=0\\\tau=0}} \frac{\partial}{\partial z}H^{(r)}\big(z,Q(z,\chi,\tau)\big)$$

*for $|\eta'|+|\delta'|\leqslant|\eta|+|\delta|-(|\alpha|+|\beta|-1)(q_0-1)$.*

**Proof.** The lemma follows from Lemma 4.4 by setting $\hat{x}=\chi$, $y=z$ and $t=\tau$ and using the family $h_\alpha(\chi,z,\tau)=\frac{\partial^{|\alpha|}}{\partial z^\alpha}H^{(r)}(z,Q(z,\chi,\tau))$, and Lemma 4.2. □

## 5. Partial jet determination

In this section, we will prove the following partial jet determination result:

**Proposition 5.1.** *Assume $M$ and $M'$ are formal generic manifolds in $\mathbb{C}^{n+d}$ of real codimension $d$ given in normal coordinates and that $M'$ is holomorphically nondegenerate. Also assume we have a formal mapping $H_0=(F_0,G_0)$ taking $(M,0)$ into $(M',0)$ satisfying* $\det\frac{\partial H_0}{\partial Z}(Z)\not\equiv 0$. *Then for any integer $k\geqslant 0$ there is an integer $N$ (depending on $k$) such that if $H$ is any formal mapping taking $M$ into $M'$ with $j_0^N(H)=j_0^N(H_0)$, then $\frac{\partial^{|\beta|}H}{\partial w^\beta}(z,0)=\frac{\partial^{|\beta|}H_0}{\partial w^\beta}(z,0)$ for $|\beta|\leqslant k$.*

Before we prove Proposition 5.1, we need Proposition 5.2 below, where the target manifold is not assumed to be holomorphically nondegenerate.

We write the power series $Q'(z, \chi, \tau)$ as

$$Q'(z', \chi', \tau') = \sum_{|\alpha|\geqslant 0} Q'_\alpha(\chi', \tau') z'^\alpha. \tag{5.1}$$

As pointed out earlier, the condition $\det \frac{\partial}{\partial z} F(z, Q(z, \chi, \tau)) \not\equiv 0$ in the proposition below is fulfilled for any mapping $H$ between the formal manifolds that satisfies the full rank condition $\det \frac{\partial H}{\partial Z}(Z) \not\equiv 0$.

**Proposition 5.2.** *For any multi-indices $\alpha$ and $\eta^0$ with $n$ components, and multi-indices $\beta$ and $\delta^0$ with $d$ components, such that $|\alpha| + |\beta| > 0$, there is a universal polynomial $R_{\alpha,\beta,\eta^0,\delta^0}((\Lambda_{\eta,\delta})_{|\eta|+|\delta|\leqslant(|\alpha|+|\beta|)(q_0+1)})$, where $q_0 = |\eta^0| + |\delta^0|$ such that the following hold: If $M$ and $M'$ are formal generic manifolds in $\mathbb{C}^{n+d}$ of real codimension $d$, given in normal coordinates by formal vector-valued functions $Q$ and $Q'$ respectively, and if $H = (F, G)$ is any formal mapping satisfying the mapping condition* (2.6) *and the following three conditions:*

$$\frac{\partial^{|\eta'|+|\delta'|}}{\partial z^{\eta'}\partial\tau^{\delta'}}\bigg|_{\substack{z=0\\ \tau=0}} \det\frac{\partial}{\partial z} F\big(z, Q(z, \chi, \tau)\big) \equiv 0, \quad |\eta'| + |\delta'| < q_0, \tag{5.2}$$

$$\frac{\partial^{|\eta^0|+|\delta^0|}}{\partial z^{\eta^0}\partial\tau^{\delta^0}}\bigg|_{\substack{z=0\\ \tau=0}} \det\frac{\partial}{\partial z} F\big(z, Q(z, \chi, \tau)\big) \not\equiv 0, \tag{5.3}$$

*and for every positive integer $k$, we have that*

$$\frac{1}{(k\eta^0)!(k\delta^0)!}\frac{\partial^{k(|\eta^0|+|\delta^0|)}}{\partial z^{k\eta^0}\partial\tau^{k\delta^0}}\bigg|_{\substack{z=0\\ \tau=0}} \lambda(z, \chi, \tau)^k = \left(\frac{1}{\eta^0!\delta^0!}\frac{\partial^{|\eta^0|+|\delta^0|}\lambda}{\partial z^{\eta^0}\partial\tau^{\delta^0}}(0, \chi, 0)\right)^k, \tag{5.4}$$

*where $\lambda(z, \chi, \tau) = \det \frac{\partial}{\partial z} F(z, Q(z, \chi, \tau))$, then we have the mapping identity*

$$\begin{aligned} &\frac{\partial^{|\beta|}}{\partial\tau^\beta}\bigg|_{\tau=0} Q'^r_\alpha\big(\bar{H}(\chi, \tau)\big) \\ &\quad = \frac{R_{\alpha,\beta,\eta^0,\delta^0}\big(\big(\frac{\partial^{|\eta|+|\delta|}}{\partial z^\eta\partial\tau^\delta}\big|_{\substack{z=0\\ \tau=0}} H^{(r)}(z, Q(z, \chi, \tau))\big)_{|\eta|+|\delta|\leqslant(|\alpha|+|\beta|)(q_0+1)}\big)}{\big(\frac{\partial^{q_0}}{\partial z^{\eta^0}\partial\tau^{\delta^0}}\big|_{\substack{z=0\\ \tau=0}} \det\frac{\partial}{\partial z} F(z, Q(z, \chi, \tau))\big)^{2(|\alpha|+|\beta|)-1}}, \end{aligned} \tag{5.5}$$

*where as before $H^{(r)}(z, w) = (F(z, w), G^r(z, w))$.*

**Remark 5.3.** For any mapping satisfying the nondegeneracy condition (2.9), there are multi-indices $\eta^0$ and $\delta^0$ such that if we set $q_0 = |\eta^0| + |\delta^0|$, we have that (5.2), (5.3) and (5.4) are satisfied. For example, we can define $\eta^0$ and $\delta^0$ by the following conditions, which correspond to a weighted lexicographical ordering of pairs of multi-indices such that the pair $(\eta^0, \delta^0)$ is the smallest pair with respect to the ordering such that (5.3) holds (other choices are also possible):

Set $\lambda(z,\chi,\tau) = \det \frac{\partial}{\partial z} F(z, Q(z,\chi,\tau))$. Let $\eta^0$ and $\delta^0$ be uniquely defined by the requirement that first (5.2) and (5.3) hold. Furthermore, we require that $\frac{\partial^{|\eta|+|\delta|}\lambda}{\partial z^\eta \partial \tau^\delta}(0,\chi,0) \equiv 0$ for all $(\eta,\delta)$ with $|\eta|+|\delta| = q_0$ and $|\eta| < |\eta^0|$, and also for all $(\eta,\delta)$ satisfying $|\eta|+|\delta| = q_0$, $|\eta| = |\eta^0|$, $\eta_k = \eta_k^0$, $k<r$ and $\eta_r > \eta_r^0$ for some $1 \leqslant r < n$. We also require $\frac{\partial^{|\eta|+|\delta|}\lambda}{\partial z^\eta \partial \tau^\delta}(0,\chi,0) \equiv 0$ for all $(\eta,\delta)$ with $|\eta|+|\delta| = q_0$, $\eta = \eta^0$, $\delta_k = \delta_k^0$, $k<r$ and $\delta_r > \delta_r^0$ for some $1 \leqslant r < d$. We then see that the condition (5.4) holds.

**Proof of Proposition 5.2.** For $|\alpha| > 0$, we use Lemma 4.5 for $\eta = (2|\alpha|+|\beta|-1)\eta^0$ and $\delta = (2|\alpha|+|\beta|-1)\delta^0$ to get that

$$\frac{\partial^{(2|\alpha|+|\beta|-1)q_0}}{\partial z^{(2|\alpha|+|\beta|-1)\eta^0} \partial \tau^{(2|\alpha|+|\beta|-1)\delta^0}}\bigg|_{\substack{z=0\\ \tau=0}} P_{\alpha,\beta}\left(\left(\frac{\partial^{|\alpha'|+|\beta'|}}{\partial z^{\alpha'} \partial \tau^{\beta'}} H^{(r)}\big(z, Q(z,\chi,\tau)\big)\right)_{\substack{1 \leqslant |\alpha'| \leqslant |\alpha| \\ |\beta'| \leqslant |\beta|}}\right)$$

is a polynomial in

$$\frac{\partial^{|\eta'|+|\gamma'|}}{\partial z^{\eta'} \partial \tau^{\gamma'}}\bigg|_{\substack{z=0\\ \tau=0}} H^{(r)}\big(z, Q(z,\chi,\tau)\big)$$

for $|\eta'| + |\delta'| \leqslant |\alpha|(q_0+1) + |\beta|$. By using property (5.4), we get from (4.30) the following identity for all $|\alpha| > 0$:

$$\begin{aligned} &\frac{\partial^{|\beta|}}{\partial \tau^\beta}\bigg|_{\tau=0} Q'_{z'^\alpha}\big(F(0,\tau), \bar{H}(\chi,\tau)\big) \\ &\quad = \frac{P'_{\alpha,\beta,\eta^0,\delta^0}\big(\big(\frac{\partial^{|\eta|+|\delta|}}{\partial z^\eta \partial \tau^\delta}\big|_{\substack{z=0\\ \tau=0}} H^{(r)}(z, Q(z,\chi,\tau))\big)_{|\eta|+|\delta| \leqslant |\alpha|(q_0+1)+|\beta|}\big)}{\big(\frac{\partial^{q_0}}{\partial z^{\eta^0} \partial \tau^{\delta^0}}\big|_{\substack{z=0\\ \tau=0}} \det \frac{\partial}{\partial z} F(z, Q(z,\chi,\tau))\big)^{2|\alpha|+|\beta|-1}}, \end{aligned} \tag{5.6}$$

where the polynomial $P'_{\alpha,\beta,\eta^0,\delta^0}$ is universal, that is, it does not depend on the manifolds $M$ and $M'$ nor on the mapping $H$. For the case $\alpha = 0$, we will instead use the identity

$$\frac{\partial^{|\beta|}}{\partial \tau^\beta}\bigg|_{\tau=0} Q'^r\big(F(0,\tau), \bar{H}(\chi,\tau)\big) = \frac{\partial^{|\beta|} G^r}{\partial w^\beta}(0,0), \tag{5.7}$$

which we get by setting $z = 0$ in (2.6) and differentiating the resulting equation with respect to $\tau$.

We will use the identities (5.6) and (5.7) to prove the proposition by induction over $|\beta|$. The case when $\beta = 0$ follows directly from the identity (5.6). Now assume that $|\beta| > 0$ and that the proposition holds for all multi-indices of lower order. We will use the observation

$$\begin{aligned} &\frac{\partial^{|\beta|}}{\partial \tau^\beta}\bigg|_{\tau=0} Q'^r_{z'^\alpha}\big(F(0,\tau), \bar{H}(\chi,\tau)\big) \\ &\quad = \frac{\partial^{|\beta|}}{\partial \tau^\beta}\bigg|_{\tau=0} Q'^r_{z'^\alpha}\big(0, \bar{H}(\chi,\tau)\big) \end{aligned}$$

$$+\sum_{\substack{\beta^1+\beta^2=\beta\\ \beta^1\neq 0}} \frac{\beta!}{\beta^1!\beta^2!}\frac{\partial^{|\beta|}}{\partial\tau^{1\beta^1}\partial\tau^{2\beta^2}}\bigg|_{\substack{\tau^1=0\\ \tau^2=0}} Q'^{r}_{z'^{\alpha}}\big(F\big(0,\tau^1\big),\bar{H}\big(\chi,\tau^2\big)\big). \tag{5.8}$$

Note that $\frac{\partial^{|\beta|}}{\partial\tau^{1\beta^1}\partial\tau^{2\beta^2}}|_{\substack{\tau^1=0\\ \tau^2=0}} Q'^{r}_{z'^{\alpha}}(F(0,\tau^1),\bar{H}(\chi,\tau^2))$ is a polynomial in $\frac{\partial^{\beta'}F}{\partial w^{\beta'}}(0)$ and $\frac{\partial^{|\beta^2|}}{\partial\tau^{2\beta^2}}|_{\tau^2=0}Q'^{r}_{z'^{\alpha'}}(0,\bar{H}(\chi,\tau^2))$ for $|\beta'|\leqslant|\beta|$ and $1\leqslant|\alpha'|\leqslant|\alpha|+|\beta^1|$. Note that $|\alpha'|+|\beta^2|\leqslant|\alpha|+|\beta|$, but $|\beta^2|<|\beta|$, so by using the induction hypothesis, we see that the proposition is true for $\beta$. $\square$

**Proposition 5.4.** *Let $M$ and $M'$ be formal generic manifolds in $\mathbb{C}^{n+d}$ of real codimension $d$, given in normal coordinates by $Q$ and $Q'$ respectively. Let $H_0=(F_0,G_0)$ be a formal mapping satisfying the mapping condition* (2.6)*, and assume that* $\det\frac{\partial}{\partial z}F_0(z,Q(z,\chi,\tau))\not\equiv 0$. *Let $q_0\geqslant 0$ be the largest integer such that $\frac{\partial^{|\eta|+|\delta|}}{\partial z^{\eta}\partial\tau^{\delta}}|_{\substack{z=0\\ \tau=0}}\det\frac{\partial}{\partial z}F_0(z,Q(z,\chi,\tau))\equiv 0$ for all $\eta$ and $\delta$ with $|\eta|+|\delta|<q_0$. Then for $|\alpha|\geqslant 0$ and $|\beta|\geqslant 0$ we have that if $H$ is any formal mapping satisfying* (2.6) *such that $j_0^N(H)=j_0^N(H_0)$, where $N=(|\alpha|+|\beta|)(q_0+1)$, then $\frac{\partial^{|\beta|}}{\partial\tau^{\beta}}|_{\tau=0}Q'_{\alpha}(\bar{H}(\chi,\tau))=\frac{\partial^{|\beta|}}{\partial\tau^{\beta}}|_{\tau=0}Q'_{\alpha}(\bar{H}_0(\chi,\tau))$.*

**Proof.** The case $\alpha=0$, $\beta=0$ follows directly from the fact that $\bar{G}(\chi,0)\equiv 0$, so we assume that $|\alpha|+|\beta|>0$. We see that the condition (5.2) for the mapping $H_0$ is fulfilled. From Remark 5.3, we see that we can pick multi-indices $\eta^0$, and $\delta^0$ with $n$ and $d$ components respectively, with $|\eta^0|+|\delta^0|=q_0$ such that the mapping $H_0$ also fulfills the conditions (5.3) and (5.4). From the assumption that $j_0^N(H)=j_0^N(H_0)$ for $N=(|\alpha|+|\beta|)(q_0+1)$, we also have that the conditions (5.2), (5.3) and (5.4) are fulfilled for the mapping $H$. It then follows that the right hand side of (5.5) for the mapping $H$ is identical to the right hand side of (5.5) for the mapping $H_0$. We conclude that the left hand sides are also identical. $\square$

We are now ready to prove Proposition 5.1.

**Proof of Proposition 5.1.** Because $M'$ is holomorphically nondegenerate, we can pick $n$ pairs $(\alpha^1,r_1),\ldots,(\alpha^n,r_n)$, where $\alpha^k$ is multi-index with $n$ components and $r_k$ is an index $1\leqslant r_k\leqslant d$, such that $\det(\frac{\partial Q'^{r_k}_{\alpha^k}}{\partial\chi^j}(\chi,\tau))_{k,j}\not\equiv 0$.

To simplify notation, let $P^k(y,t)=Q'^{r_k}_{\alpha^k}(y,t)$ for $1\leqslant k\leqslant n$ and $P^k(y,t)=Q'^{k-n}_0(y,t)$ for $n+1\leqslant k\leqslant n+d$. Let $Y=(y,t)$ and $P(Y)=(P^1(Y),\ldots,P^{n+d}(Y))$. Furthermore, let $b=\max_{1\leqslant k\leqslant n}|\alpha^k|$.

Note that $Q'_0(y,t)=t$, so

$$\det\frac{\partial P}{\partial Y}(y,t)=\det\left(\frac{\partial Q'^{r_k}_{\alpha^k}}{\partial\chi^j}(y,t)\right)_{k,j}\not\equiv 0. \tag{5.9}$$

Define

$$\lambda(Y):=\det\frac{\partial P}{\partial Y}(Y). \tag{5.10}$$

As noted above, $\lambda(Y) \not\equiv 0$, and from the assumption that $H_0$ is of generically full rank, we also have that $\lambda(\bar{H}_0(\chi, \tau)) \not\equiv 0$. Let $B_1$ be an integer and $\beta^1$ a multi-index with $d$ components such that there exists a multi-index $\eta$ such that $|\eta| + |\beta^1| = B_1$ and $\frac{\partial^{B_1}}{\partial \chi^\eta \tau^{\beta^1}}\big|_{\substack{\chi=0\\ \tau=0}} \lambda(\bar{H}_0(\chi, \tau)) \neq 0$.

Now, let the integer $k \geqslant 0$. We define the bound $N_k^1$ by

$$N_k^1 = \max\bigl\{\bigl(b + \bigl|\beta^1\bigr| + k\bigr)(q_0 + 1), B_1\bigr\}. \tag{5.11}$$

Assume that

$$j_0^{N_k^1}(H) = j_0^{N_k^1}(H_0). \tag{5.12}$$

We see from Proposition 5.4 that

$$\frac{\partial^{|\beta|}}{\partial \tau^\beta}\bigg|_{\tau=0} Q'_\alpha\bigl(\bar{H}(\chi, \tau)\bigr) = \frac{\partial^{|\beta|}}{\partial \tau^\beta}\bigg|_{\tau=0} Q'_\alpha\bigl(\bar{H}_0(\chi, \tau)\bigr)$$

for $|\alpha| \leqslant b$, $|\beta| \leqslant |\beta^1| + k$. Thus,

$$\frac{\partial^{|\beta|}}{\partial \tau^\beta}\bigg|_{\tau=0} P\bigl(\bar{H}(\chi, \tau)\bigr) = \frac{\partial^{|\beta|}}{\partial \tau^\beta}\bigg|_{\tau=0} P\bigl(\bar{H}_0(\chi, \tau)\bigr), \quad |\beta| \leqslant \bigl|\beta^1\bigr| + k. \tag{5.13}$$

Now from Proposition 3.1, it follows that $\frac{\partial^{|\beta|} H}{\partial w^\beta}(z, 0) = \frac{\partial^{|\beta|} H_0}{\partial w^\beta}(z, 0)$ for $|\beta| \leqslant k$. $\square$

## 6. The finite type case

In this section, we will prove Theorems 1.3 and 1.4, but first we need to prove Proposition 6.2, which is a generalization of Proposition 5.1.

Assume $M$ is a formal generic manifold in $\mathbb{C}^{n+d}$ of real codimension $d$ given in normal coordinates by $Q$. For $j \geqslant 1$, we define the formal mappings $U^j : \mathbb{C}^{nj} \times \mathbb{C}^d \to \mathbb{C}^d$ by the following formulas, where the variables $t \in C^d$ and $z, \chi, z^k, \chi^k \in \mathbb{C}^n$ for all $k$:

$$U^1(z; t) = t, \tag{6.1}$$

$$U^{2j}\bigl(z, \chi, z^2, \chi^2, \ldots, z^j, \chi^j; t\bigr) = U^{2j-1}\bigl(z, \chi, z^2, \chi^2, \ldots, z^j; Q\bigl(z^j, \chi^j, t\bigr)\bigr), \quad j \geqslant 1 \tag{6.2}$$

and

$$\begin{aligned} &U^{2j+1}\bigl(z, \chi, z^2, \chi^2, \ldots, z^j, \chi^j, z^{j+1}; t\bigr) \\ &\quad = U^{2j}\bigl(z, \chi, z^2, \chi^2, \ldots, z^j, \chi^j; \bar{Q}\bigl(\chi^j, z^{j+1}, t\bigr)\bigr), \quad j \geqslant 1. \end{aligned} \tag{6.3}$$

Furthermore, let the formal mappings $S^j\colon \mathbb{C}^{nj} \times \mathbb{C}^d \to \mathbb{C}^n \times \mathbb{C}^d$ for $j \geqslant 1$ be defined by

$$S^j\bigl(z, \chi, z^2, \chi^2, \ldots; t\bigr) = \bigl(z, U^j\bigl(z, \chi, z^2, \chi^2, \ldots; t\bigr)\bigr). \tag{6.4}$$

**Remark 6.1.** By setting $t = 0$ in $S^j$, i.e., $S^j(z, \chi, z^2, \chi^2, \dots; 0)$, we get the usual iterated Segre varieties. However, we need to take derivatives with respect to $t$ before setting $t = 0$, so we will keep the above notation.

We will also use the complex conjugates $\bar{U}^j(\chi, z, \chi^2, \dots; t)$ and $\bar{S}^j(\chi, z, \chi^2, \dots; t)$. From the reality condition (2.4), we have

$$\bigl(z, Q\bigl(z, \chi, \bar{U}^{j+1}\bigl(\chi, z, \chi^2, z^2, \dots; t\bigr)\bigr)\bigr) = S^j\bigl(z, \chi^2, z^2, \dots; t\bigr), \quad j \geqslant 1. \tag{6.5}$$

By plugging $\tau = \bar{U}^{j+1}(\chi, z, \chi^2, \dots; t)$ into the mapping equation (2.6) and using the identity above, we get

$$Q'\bigl(F\bigl(S^j\bigl(z, \chi^2, z^2, \dots; t\bigr)\bigr), \bar{H}\bigl(\bar{S}^{j+1}\bigl(\chi, z, \chi^2, \dots; t\bigr)\bigr)\bigr) = G\bigl(S^j\bigl(z, \chi^2, z^2, \dots; t\bigr)\bigr). \tag{6.6}$$

We will prove the following result:

**Proposition 6.2.** *Assume that $H_0$ is a formal mapping of generically full rank between two formal generic manifolds $M$ and $M'$ given in normal coordinates, where $M'$ is holomorphically nondegenerate. Then for all $j \geqslant 1$ and $k \geqslant 0$, there exist an integer $N_k^j$ such that if $H$ is any formal mapping between $M$ and $M'$ satisfying $j_0^{N_k^j}(H) = j_0^{N_k^j}(H_0)$, then*

$$\frac{\partial^{|\beta|}}{\partial t^\beta}\bigg|_{t=0}\bigl(H \circ S^j\bigr) \equiv \frac{\partial^{|\beta|}}{\partial t^\beta}\bigg|_{t=0}\bigl(H_0 \circ S^j\bigr) \tag{6.7}$$

*for $|\beta| \leqslant k$.*

Explicit expressions for the bounds $N_k^j$ are given by (5.11) and by (6.25) in the proof below.

Before we prove Proposition 6.2, we need some more mapping identities.

**Lemma 6.3.** *Let the polynomials $P_{\alpha,\beta}$ be as defined in Section* 4. *Let $M$ and $M'$ be formal generic manifolds in $\mathbb{C}^{n+d}$ of real codimension $d$ given in normal coordinates by $Q$ and $Q'$ respectively. Assume $H$ is a formal mapping satisfying the mapping condition* (2.6) *and the nondegeneracy condition* (2.9). *Then for any integer $j \geqslant 1$ and multi-indices $\alpha$ and $\beta$ with $n$ and $d$ components respectively with $|\alpha| > 0$, we have the mapping identity*

$$\begin{aligned}&\frac{\partial^{|\beta|}}{\partial t^\beta} Q'^r_{z'^\alpha}\bigl(F\bigl(S^j\bigl(z, \chi^2, z^2, \dots; t\bigr)\bigr), \bar{H}\bigl(\bar{S}^{j+1}\bigl(\chi, z, \chi^2, \dots; t\bigr)\bigr)\bigr)\\ &\quad = \frac{P_{\alpha,\beta}\bigl(\bigl(\frac{\partial^{|\beta'|}}{\partial t^{\beta'}}\bigl(\bigl(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}} H^{(r)}(z, Q(z, \chi, \tau))\bigr)\bigr|_{\tau=\bar{U}^{j+1}(\chi, z, \chi^2, \dots; t)}\bigr)\bigr)_{\substack{1 \leqslant |\alpha'| \leqslant |\alpha| \\ |\beta'| \leqslant |\beta|}}\bigr)}{\det\bigl(\bigl(\frac{\partial}{\partial z} F(z, Q(z, \chi, \tau))\bigr)\bigr|_{\tau=\bar{U}^{j+1}(\chi, z, \chi^2, \dots; t)}\bigr)^{2|\alpha|+|\beta|-1}}.\end{aligned} \tag{6.8}$$

*Furthermore, let $q_{j+1}$ be the largest integer such that*

$$\frac{\partial^{|\beta'|}}{\partial t^{\beta'}}\bigg|_{t=0} \det\biggl(\biggl(\frac{\partial}{\partial z} F\bigl(z, Q(z, \chi, \tau)\bigr)\biggr)\bigg|_{\tau=\bar{U}^{j+1}(\chi, z, \chi^2, \dots; t)}\biggr) \equiv 0$$

*for all* $|\beta'| < q_{j+1}$. *We have that*

$$\frac{\partial^{|\delta|}}{\partial t^{\delta}}\bigg|_{t=0} P_{\alpha,\beta}\left(\left(\frac{\partial^{|\beta'|}}{\partial t^{\beta'}}\left(\left(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}}H^{(r)}\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)\right)_{\substack{1\leqslant|\alpha'|\leqslant|\alpha|\\|\beta'|\leqslant|\beta|}}\right)$$

*is a polynomial in*

$$\frac{\partial^{|\delta'|}}{\partial t^{\delta'}}\bigg|_{t=0}\left(\left(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}}H^{(r)}\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)$$

*for* $1 \leqslant |\alpha'| \leqslant |\alpha|$ *and* $|\delta'| \leqslant |\delta| - |\beta|(q_{j+1} - 1)$.

**Proof.** By setting $\tau = \bar{U}^{j+1}(\chi, z, \chi^2, \dots; t)$ in the identity (4.14) and using (6.5), we get

$$\begin{aligned} &Q'^{r}_{z'^{\alpha}}\big(F\big(S^j\big(z,\chi^2,z^2,\dots;t\big)\big), \bar{H}\big(\bar{S}^{j+1}\big(\chi,z,\chi^2,\dots;t\big)\big)\big) \\ &\quad = \frac{P_\alpha(((\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}}H^{(r)}(z,Q(z,\chi,\tau)))|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)})_{1\leqslant|\alpha'|\leqslant|\alpha|})}{\det((\frac{\partial}{\partial z}F(z,Q(z,\chi,\tau)))|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)})^{k_\alpha}}, \end{aligned} \tag{6.9}$$

where, as before, $k_\alpha = 2|\alpha| - 1$. Now using

$$h_{\alpha'} = \left(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}}H^{(r)}\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}$$

in Lemma 4.4 with $x = (\chi, z, \chi^2, \dots)$ (not including $t$) and with no variable $y$, we get (6.3).

From the second part of Lemma 4.4, we get that

$$\frac{\partial^{|\delta|}}{\partial t^{\delta}}\bigg|_{t=0} P_{\alpha,\beta}\left(\left(\frac{\partial^{|\beta'|}}{\partial t^{\beta'}}\left(\left(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}}H^{(r)}\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)\right)_{\substack{1\leqslant|\alpha'|\leqslant|\alpha|\\|\beta'|\leqslant|\beta|}}\right)$$

is a polynomial in

$$\frac{\partial^{|\delta^1|}}{\partial t^{\delta^1}}\bigg|_{t=0} P_{\alpha}\left(\left(\left(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}}H^{(r)}\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)_{1\leqslant|\alpha'|\leqslant|\alpha|}\right)$$

and

$$\frac{\partial^{|\delta^2|}}{\partial t^{\delta^2}}\bigg|_{t=0} \det\left(\left(\frac{\partial}{\partial z}F\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)$$

for

$$\big|\delta^1\big| \leqslant |\delta| - |\beta|(q_{j+1} - 1) \quad \text{and} \quad \big|\delta^2\big| \leqslant |\delta| - (k_\alpha - 1)q_{j+1} - |\beta|(q_{j+1} - 1).$$

We also see directly that for each $\delta^1$ that

$$\frac{\partial^{|\delta^1|}}{\partial t^{\delta^1}}\bigg|_{t=0} P_\alpha\left(\left(\left(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}}H^{(r)}\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)_{1\leqslant|\alpha'|\leqslant|\alpha|}\right)$$

is a polynomial in

$$\frac{\partial^{|\delta'|}}{\partial t^{\delta'}}\bigg|_{t=0}\left(\left(\frac{\partial^{|\alpha'|}}{\partial z^{\alpha'}}H^{(r)}\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)$$

for $1\leqslant|\alpha'|\leqslant|\alpha|$ and $|\delta'|\leqslant|\delta^1|$. By combining these results, we get the conclusion of the second part of the lemma. $\square$

We can now prove our final universal mapping identity:

**Proposition 6.4.** *For any multi-index $\alpha$ with $n$ components, and multi-indices $\beta$ and $\gamma$ with $d$ components, such that $|\alpha|+|\beta|>0$, there is a universal polynomial*

$$R_{\alpha,\beta,\gamma}\big((\Lambda_{\eta,\delta})_{|\eta|+|\delta|\leqslant(|\alpha|+|\beta|)(2|\gamma|+1)-|\gamma|}\big)$$

*such that the following hold: If $M$ and $M'$ are formal generic submanifolds in $\mathbb{C}^{n+d}$ of real codimension $d$ given in normal coordinates by $Q$ and $Q'$ respectively. If $j\geqslant 1$ and if $H=(F,G)$ is any formal mapping satisfying the mapping condition* (2.6) *and the following three conditions:*

$$\frac{\partial^{|\delta'|}}{\partial t^{\delta'}}\bigg|_{t=0}\det\left(\left(\frac{\partial}{\partial z}F\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)\equiv 0,\quad |\delta'|<|\gamma|, \tag{6.10}$$

$$\frac{\partial^{|\gamma|}}{\partial t^{\gamma}}\bigg|_{t=0}\det\left(\left(\frac{\partial}{\partial z}F\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)\not\equiv 0, \tag{6.11}$$

*and for every positive integer $k$, we have that*

$$\begin{aligned}&\frac{1}{(k\gamma)!}\frac{\partial^{k(|\gamma|)}}{\partial t^{k\gamma}}\bigg|_{t=0}\det\left(\left(\frac{\partial}{\partial z}F\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)^k\\ &\quad=\left(\frac{1}{\gamma!}\frac{\partial^{|\gamma|}}{\partial t^{\gamma}}\bigg|_{t=0}\det\left(\left(\frac{\partial}{\partial z}F\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)\right)^k,\end{aligned} \tag{6.12}$$

*then for each $r$ in $\{1,\dots,d\}$, we have the mapping identity*

$$\begin{aligned}&\frac{\partial^{|\beta|}}{\partial t^{\beta}}\bigg|_{t=0} Q'^{r}_{z'^\alpha}\big(F\big(S^j\big(z,\chi^2,z^2,\dots,0\big)\big), \bar{H}\big(\bar{S}^{j+1}\big(\chi,z,\chi^2,\dots;t\big)\big)\big)\\ &\quad=\frac{R_{\alpha,\beta,\gamma}\big(\big(\frac{\partial^{|\delta|}}{\partial t^{\delta}}|_{t=0}h^{r,j}_{\eta}(\chi,z,\dots;t)\big)_{|\eta|+|\delta|\leqslant(|\alpha|+|\beta|)(2|\gamma|+1)-|\gamma|}\big)}{\big(\frac{1}{\gamma!}\frac{\partial^{|\gamma|}}{\partial t^{\gamma}}|_{t=0}\det\big(\big(\frac{\partial}{\partial z}F(z,Q(z,\chi,\tau))\big)\big)|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\big)\big)^{2(|\alpha|+|\beta|)-1}},\end{aligned} \tag{6.13}$$

*where* $h_\eta^{r,j}(\chi, z, \ldots; t) = (\frac{\partial^{|\eta|}}{\partial z^\eta} H^{(r)}(z, Q(z, \chi, \tau)))|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\ldots;t)}$ *and, as before,* $H^{(r)}(z, w) = (F(z, w), G^r(z, w))$.

**Remark 6.5.** For any mapping satisfying the nondegeneracy condition (2.9), there is a multi-index $\gamma$ such that (6.10), (6.11) and (6.12) are satisfied. For example, a similar ordering of multi-indices as in Remark 5.3 can be used.

**Proof of Proposition 6.4.** For $|\alpha| > 0$, we use Lemma 6.3 for $\delta = (2|\alpha| + |\beta| - 1)\gamma$ to get that

$$\frac{\partial^{(2|\alpha|+|\beta|-1)|\gamma|}}{\partial t^{(2|\alpha|+|\beta|-1)\gamma}}\bigg|_{t=0} P_{\alpha,\beta}\left(\left(\frac{\partial^{|\beta'|}}{\partial t^{\beta'}} h_{\alpha'}^{r,j}(\chi, z, \ldots; t)\right)_{\substack{1\leqslant|\alpha'|\leqslant|\alpha| \\ |\beta'|\leqslant|\beta|}}\right)$$

is a polynomial in

$$\frac{\partial^{|\delta'|}}{\partial t^{\delta'}}\bigg|_{t=0} h_{\alpha'}^{r,j}(\chi, z, \ldots; t)$$

for $|\delta'| \leqslant (2|\alpha| - 1)|\gamma| + |\beta|$ and $1 \leqslant |\alpha'| \leqslant |\alpha|$. By using property (6.12), we get from (6.8) the following identity for all $|\alpha| > 0$:

$$\begin{aligned}
&\frac{\partial^{|\beta|}}{\partial t^\beta}\bigg|_{t=0} Q'^{r}_{z'^\alpha}\big(F\big(S^j\big(z, \chi^2, z^2, \ldots; t\big)\big), \bar{H}\big(\bar{S}^{j+1}\big(\chi, z, \chi^2, \ldots; t\big)\big)\big) \\
&\quad = \frac{P'_{\alpha,\beta,\gamma}((\frac{\partial^{|\delta|}}{\partial t^\delta}|_{t=0} h_\eta^{r,j}(\chi, z, \ldots; t))_{|\eta|+|\delta|\leqslant(2|\alpha|-1)|\gamma|+|\alpha|+|\beta|})}{(\frac{1}{\gamma!}\frac{\partial^{|\gamma|}}{\partial t^\gamma}|_{t=0} \det((\frac{\partial}{\partial z} F(z, Q(z, \chi, \tau)))|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\ldots;t)}))^{2|\alpha|+|\beta|-1}},
\end{aligned}$$

where the polynomial $P'_{\alpha,\beta,\gamma}$ is universal, that is, it does not depend on the manifolds $M$ and $M'$ nor the mapping $H$. For the case $\alpha = 0$, we will use the identity

$$\begin{aligned}
&\frac{\partial^{|\beta|}}{\partial t^\beta}\bigg|_{t=0} Q'^{r}\big(F\big(S^j\big(z, \chi^2, z^2, \ldots; t\big)\big), \bar{H}\big(\bar{S}^{j+1}\big(\chi, z, \chi^2, \ldots; t\big)\big)\big) \\
&\quad = \frac{\partial^{|\beta|}}{\partial t^\beta}\bigg|_{t=0} G^r\big(S^j\big(z, \chi^2, z^2, \ldots; t\big)\big),
\end{aligned}$$

which we get by differentiating (6.6).

The proposition now follows by induction over $|\beta|$ using the observation

$$\begin{aligned}
&\frac{\partial^{|\beta|}}{\partial t^\beta}\bigg|_{t=0} y_\alpha^{r,j}(\chi, z, \ldots; t, t) \\
&\quad = \sum_{\beta^1+\beta^2=\beta} \frac{\beta!}{\beta^1!\beta^2!} \frac{\partial^{|\beta|}}{\partial t^{1\beta^1} \partial t^{2\beta^2}}\bigg|_{\substack{t^1=0 \\ t^2=0}} y_\alpha^{r,j}\big(\chi, z, \ldots; t^1, t^2\big),
\end{aligned} \tag{6.14}$$

where

$$y_\alpha^{r,j}\big(\chi, z, \ldots; t^1, t^2\big) = Q'^{r}_{z'^\alpha}\big(F\big(S^j\big(z, \chi^2, z^2, \ldots; t^1\big)\big), \bar{H}\big(\bar{S}^{j+1}\big(\chi, z, \chi^2, \ldots; t^2\big)\big)\big). \qquad \square$$

We also need the following lemma:

**Lemma 6.6.** *Let $h(z,w)$ and $h_0(z,w)$ be power series. If for some positive integers $N$, $j$ and nonnegative integer $k$, we have that*

$$\frac{\partial^{|\alpha|}h}{\partial Z^\alpha}\bigl(z, U^j(z,\chi,\dots;t)\bigr) = \frac{\partial^{|\alpha|}h_0}{\partial Z^\alpha}\bigl(z, U^j(z,\chi,\dots;t)\bigr) + O\bigl(|t|^N\bigr) \tag{6.15}$$

*for $|\alpha| \leqslant k$ then*

$$\frac{\partial^{|\alpha|}h}{\partial Z^\alpha}\bigl(z, U^j(z,\chi,\dots;t)\bigr) = \frac{\partial^{|\alpha|}h_0}{\partial Z^\alpha}\bigl(z, U^j(z,\chi,\dots;t)\bigr) + O\bigl(|t|^{N-1}\bigr) \tag{6.16}$$

*for $|\alpha| \leqslant k+1$.*

**Proof.** By applying the vector field $\frac{\partial}{\partial t}$ to Eq. (6.15), we get that

$$\begin{aligned}
&\frac{\partial^{|\alpha|+1}h}{\partial Z^\alpha \partial w}\bigl(z, U^j(z,\chi,\dots;t)\bigr)\frac{\partial U^j}{\partial t}(z,\chi,\dots;t) \\
&\quad = \frac{\partial^{|\alpha|+1}h_0}{\partial Z^\alpha \partial w}\bigl(z, U^j(z,\chi,\dots;t)\bigr)\frac{\partial U^j}{\partial t}(z,\chi,\dots;t) + O\bigl(|t|^{N-1}\bigr)
\end{aligned}$$

for $|\alpha| \leqslant k$. Now since $\frac{\partial U^j}{\partial t}(z,\chi,\dots;t)$ is an invertible matrix in the ring of formal power series, we have that

$$\frac{\partial^{|\alpha|+1}h}{\partial Z^\alpha \partial w}\bigl(z, U^j(z,\chi,\dots;t)\bigr) = \frac{\partial^{|\alpha|+1}h_0}{\partial Z^\alpha \partial w}\bigl(z, U^j(z,\chi,\dots;t)\bigr) + O\bigl(|t|^{N-1}\bigr) \tag{6.17}$$

for $|\alpha| \leqslant k$.

Now, by applying the vector field $\frac{\partial}{\partial z}$ to Eq. (6.15), we get

$$\begin{aligned}
&\frac{\partial^{|\alpha|+1}h}{\partial Z^\alpha \partial z}\bigl(z, U^j(z,\chi,\dots;t)\bigr) + \frac{\partial^{|\alpha|+1}h}{\partial Z^\alpha \partial w}\bigl(z, U^j(z,\chi,\dots;t)\bigr)\frac{\partial U^j}{\partial z}(z,\chi,\dots;t) \\
&\quad = \frac{\partial^{|\alpha|+1}h_0}{\partial Z^\alpha \partial z}\bigl(z, U^j(z,\chi,\dots;t)\bigr) + \frac{\partial^{|\alpha|+1}h}{\partial Z^\alpha \partial w}\bigl(z, U^j(z,\chi,\dots;t)\bigr)\frac{\partial U^j}{\partial z}(z,\chi,\dots;t) \\
&\qquad + O\bigl(|t|^N\bigr)
\end{aligned}$$

for $|\alpha| \leqslant k$. By using (6.17), we get

$$\frac{\partial^{|\alpha|+1}h}{\partial Z^\alpha \partial z}\bigl(z, U^j(z,\chi,\dots;t)\bigr) = \frac{\partial^{|\alpha|+1}h_0}{\partial Z^\alpha \partial z}\bigl(z, U^j(z,\chi,\dots;t)\bigr) + O\bigl(|t|^{N-1}\bigr) \tag{6.18}$$

for $|\alpha| \leqslant k$.

Eqs. (6.17) and (6.18) prove the lemma. □

**Corollary 6.7.** *Let $h(z,w)$ and $h_0(z,w)$ be power series. If for some positive integer $j$ and nonnegative integer $k$, we have that*

$$\frac{\partial^{|\beta|}}{\partial t^{\beta}}\bigg|_{t=0} h\big(S^j(z,\chi,\ldots;t)\big) = \frac{\partial^{|\beta|}}{\partial t^{\beta}}\bigg|_{t=0} h_0\big(S^j(z,\chi,\ldots;t)\big) \tag{6.19}$$

*for $|\beta| \leqslant k$, then*

$$\frac{\partial^{|\beta|}}{\partial t^{\beta}}\bigg|_{t=0} \frac{\partial^{|\alpha|}h}{\partial Z^{\alpha}}\big(S^j(z,\chi,\ldots;t)\big) = \frac{\partial^{|\beta|}}{\partial t^{\beta}}\bigg|_{t=0} \frac{\partial^{|\alpha|}h_0}{\partial Z^{\alpha}}\big(S^j(z,\chi,\ldots;t)\big) \tag{6.20}$$

*for $|\alpha| + |\beta| \leqslant k$.*

**Proof of Proposition 6.2.** From Proposition 5.1, we see that the proposition is true for $j = 1$. In the proof of Proposition 5.1, we gave an explicit expression for the bound $N_k^1$ by Eq. (5.11).

We will prove the proposition by induction over $j$, so assume for some $j \geqslant 1$ that we have defined the integers $N_k^j$ such that (6.7) holds. We define $q_{j+1}$ to be the largest integer such that

$$\frac{\partial^{|\beta'|}}{\partial t^{\beta'}}\bigg|_{t=0} \det\left(\left(\frac{\partial}{\partial z} F_0\big(z, Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\ldots;t)}\right) \equiv 0$$

for all $|\beta'| < q_{j+1}$. We also define the integers $N_{|\alpha|,k}^{j+1}$ by

$$N_{0,0}^{j+1} = N_0^j, \qquad N_{|\alpha|,k}^{j+1} = N^j_{r_{\alpha,k}^{j+1}}, \quad \text{for } |\alpha| + k > 0,$$

where $r_{\alpha,k}^{j+1} = (|\alpha| + k)(2q_{j+1} + 1) - q_{j+1}$.

We start by proving the following:

**Lemma 6.8.** *Let $M$, $M'$ and $H_0$ be as above. If $H$ is a formal mapping satisfying the mapping condition* (2.6) *and the condition $j_0^{N_{|\alpha|,k}^{j+1}}(H) = j_0^{N_{|\alpha|,k}^{j+1}}(H_0)$ then*

$$\begin{aligned} &\frac{\partial^{|\beta|}}{\partial t^{\beta}}\bigg|_{t=0} Q'^{r}_{z'^{\alpha}}\big(F_0\big(S^j\big(z,\chi^2,z^2,\ldots;0\big)\big), \bar{H}\big(\bar{S}^{j+1}\big(\chi,z,\chi^2,\ldots;t\big)\big)\big) \\ &\quad= \frac{\partial^{|\beta|}}{\partial t^{\beta}}\bigg|_{t=0} Q'^{r}_{z'^{\alpha}}\big(F_0\big(S^j\big(z,\chi^2,z^2,\ldots;0\big)\big), \bar{H}_0\big(\bar{S}^{j+1}\big(\chi,z,\chi^2,\ldots;t\big)\big)\big) \end{aligned} \tag{6.21}$$

*for $|\beta| \leqslant k$.*

**Proof.** The case when $|\alpha| + k = 0$ follows directly from the identity (6.6) and the induction hypothesis. For the case $|\alpha| + k > 0$, the lemma is a corollary of Proposition 6.4: Pick a multi-index $\gamma$ with $|\gamma| = q_{j+1}$ such that the conditions (6.10), (6.11) and (6.12) are fulfilled for the mapping $H_0$. Since $|\alpha| + k > 0$, we have that $(|\alpha| + k)(2q_{j+1} + 1) - q_{j+1} \geqslant q_{j+1} + 1$. Therefore $N_{\alpha,k}^{j+1} \geqslant N^j_{q_{j+1}+1}$. We therefore have that $F(S^j(z,\chi^2,z^2,\ldots;0)) = F_0(S^j(z,\chi^2,z^2,\ldots;0))$, and from Corollary 6.7, we have that

$$\begin{aligned}&\frac{\partial^{|\gamma|}}{\partial t^{\gamma}}\bigg|_{t=0}\det\left(\left(\frac{\partial}{\partial z}F\big(z,Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right)\\&\quad=\frac{\partial^{|\gamma|}}{\partial t^{\gamma}}\bigg|_{t=0}\det\left(\left(\frac{\partial}{\partial z}F_0\big(z,Q(z,\chi,\tau)\big)\right)\bigg|_{\tau=\bar{U}^{j+1}(\chi,z,\chi^2,\dots;t)}\right).\end{aligned}$$

Now for a $\beta$ with $|\beta|<k$, we observe that the mapping $H$ also satisfies the conditions (6.10), (6.11) and (6.12) and that both mappings $H$ and $H_0$ satisfy the mapping identity (6.13). We have already observed that the denominator of the right hand side of (6.13) are the same for the two mappings and again from Corollary 6.7, we see that also the numerators are the same. Because the right hand sides are identical also the left hand sides will agree. The lemma now follows from the observation above that $F(S^j(z,\chi^2,z^2,\dots;0))=F_0(S^j(z,\chi^2,z^2,\dots;0))$. $\square$

Now consider the pairs $(\alpha^p,r_p)$ as defined in the previous section, and define $P_{j+1}:\mathbb{C}^{(j+1)n}\times\mathbb{C}^{n+d}\to\mathbb{C}^{n+d}$ by

$$P_{j+1}\big(\chi,z,\chi^2,z^2,\dots,Y\big)=\begin{pmatrix}Q'^{r_1}_{z'^{\alpha^1}}(F_0(S^j(z,\chi^2,z^2,\dots;0)),Y)\\ \vdots\\ Q'^{r_n}_{z'^{\alpha^n}}(F_0(S^j(z,\chi^2,z^2,\dots;0)),Y)\\ Q'^1(F_0(S^j(z,\chi^2,z^2,\dots;0)),Y)\\ \vdots\\ Q'^d(F_0(S^j(z,\chi^2,z^2,\dots;0)),Y)\end{pmatrix}. \tag{6.22}$$

We observe that

$$\det\frac{\partial P_{j+1}}{\partial Y}\big(\chi,z,\chi^2,\dots,\bar{H}_0\big(\bar{S}^{j+1}\big(\chi,z,\chi^2,\dots;t\big)\big)\big)\not\equiv 0. \tag{6.23}$$

This is true, because if we set $z=0$ and $\chi^p=0$, $z^p=0$ for $p\geqslant 2$, in the left hand side of the above equation, we get

$$\det\left(\frac{\partial Q'^{r_p}_{z'^{\alpha^p}}}{\partial\chi^k}\big(0,\bar{H}_0(\chi,t)\big)\right)_{p,k},$$

which is not identically zero.

Let $B_{j+1}$ be an integer and $\beta^{j+1}$ a multi-index with $d$ components such that there exist multi-indices $\eta^k$, $1\leqslant k\leqslant j+1$, where each multi-index $\eta^k$ has $n$ components, and we have $|\eta^1|+\dots+|\eta^{j+1}|+|\beta^{j+1}|=B_{j+1}$ and

$$\left(\frac{\partial^{B_{j+1}}}{\partial\chi^{\eta^1}\dots\partial t^{\beta^{j+1}}}\det\frac{\partial P_{j+1}}{\partial Y}\big(\chi,z,\chi^2,\dots,\bar{H}_0\big(\bar{S}^{j+1}\big(\chi,z,\chi^2,\dots;t\big)\big)\big)\right)(0)\neq 0. \tag{6.24}$$

Let

$$N_k^{j+1}=\max\big\{N^j_{r_{k,j}},B_{j+1}\big\}, \tag{6.25}$$

where $r_{k,j} = (b + |\beta^{j+1}| + k)(2q_{j+1} + 1) - q_{j+1}$, where $b = \max_p\{|\alpha^p|\}$ as in the previous section, and where $B_{j+1}$ is defined above. Now from Proposition 3.1 and Lemma 6.8, it follows that if

$$j_0^{N_k^{j+1}}(H) = j_0^{N_k^{j+1}}(H_0),$$

then

$$\frac{\partial^{|\beta|}}{\partial t^\beta}\bigg|_{t=0} \bar{H}\big(\bar{S}^{j+1}(\chi, z, \chi^2, \ldots; t)\big) = \frac{\partial^{|\beta|}}{\partial t^\beta}\bigg|_{t=0} \bar{H}_0\big(\bar{S}^{j+1}(\chi, z, \chi^2, \ldots; t)\big),$$

for $|\beta| \leqslant k$, so we have proved that the proposition holds for $j + 1$. □

**Remark 6.9.** Note that we can take $B_{j+1} = B_1$ and $\beta^{j+1} = \beta^1$. Furthermore, we note that $p_2 \leqslant p_0$ and that $q_{j+1} \leqslant p_j$ for $j \geqslant 2$, so we can use $p_0$ for an upper bound. With these choices, we get a simpler expression for the bound of the jet, but we get, in general, a better estimate if we are more flexible in how we choose these constants.

**Proof of Theorem 1.3.** Now, if $M$ is a generic manifold of finite type at 0, we know that the $(d+1)$th Segre map $(\xi^1, \ldots, \xi^d) \mapsto S^{d+1}(\xi^1, \ldots, \xi^d, 0)$ is generically of full rank, see [1] or [5]. Therefore, it follows that if $j_0^{N_0^{d+1}}(H) = j_0^{N_0^{d+1}}(H_0)$, then $H(z, w) \equiv H_0(z, w)$. □

**Proof of Theorem 1.4.** The theorem follows from the proof of Proposition 5.1 and Proposition 6.2 by inspecting how the integers $N_k^j$ change when we move away from a point $p_0 \in M \cap U$. We will show that if the point $p \in M$ is sufficiently close to $p_0$, then $N_0^{d+1}(p) \leqslant N_0^{d+1}(p_0)$, and from this the theorem follows.

Take $p_0 \in M \cap U$ and let $Q(z, \chi, \tau)$ represent a choice of normal coordinates at $p_0$ and $Q'(z', \chi', \tau')$ a choice of normal coordinates for $M'$ at $H(p_0)$. Then for all $p$ in a neighborhood of $p_0$ in $M \cap U$ we can pick normal coordinates for $M$ at $p$ represented by $Q_p(z, \chi, \tau)$ such that $Q_{p_0}(z, \chi, \tau) \equiv Q(z, \chi, \tau)$ and so that all the coefficients of $Q_p(z, \chi, \tau)$ as a vector-valued power series are continuous as a function of $p$. (This is done by taking the standard transformations, as described in [3], that take $M$ given in normal coordinates at $p_0$ by $Q(z, \chi, \tau)$ into normal coordinates at $p$. The coefficients of $Q_p(z, \chi, \tau)$ will in fact depend real-analytically on $p$, but we will only use continuity in this paper. See also [18], Lemma 4.1.) We pick a similar family $Q'_{p'}(z, \chi, \tau)$ representing normal coordinates of $M'$ in a neighborhood of $H_0(p_0)$. Because our original map $H_0$ is continuous the coefficients of $Q'_{H_0(p)}(z', \chi', \tau')$ also depend continuously on $p$, for $p$ sufficiently close to $p_0$. Let $H_{0,p}(z, w)$ denote the mapping $H_0$ after the transformation into these choices of normal coordinates at $p \in M$ and $H_0(p) \in M'$ respectively. With this choice of transformations into normal coordinates, also the coefficients of $H_{0,p}(z, w)$ are continuous functions of $p$.

We start by looking how the integer $q_0$ change with the point $p$. If $\eta$, $\delta$ are multi-indices such that

$$\frac{\partial^{|\eta|+|\delta|}}{\partial z^\eta \partial \tau^\delta}\bigg|_{\substack{z=0\\ \tau=0}} \det \frac{\partial}{\partial z} F_{0,p}\big(z, Q(z, \chi, \tau)\big) \not\equiv 0 \tag{6.26}$$

for the point $p = p_0$, then, because of continuity of the coefficients, we have that (6.26) holds for all points $p$ sufficiently close to $p_0$. We conclude that

$$q_0(p) \leqslant q_0(p_0).$$

Next, for any $n$ pairs $(\alpha^j, r_j)$, $1 \leqslant j \leqslant n$ such that

$$\det\left(\frac{\partial {Q'_{H_0(p)}}^{r_k}_{\alpha^k}}{\partial \chi^j}(\chi, \tau)\right)_{k,j} \not\equiv 0 \tag{6.27}$$

for the point $p = p_0$, we have again from continuity that the same condition holds for all points $p$ sufficiently close to $p_0$, so the same choice of pairs work for $p$ as for $p_0$, if $p$ is sufficiently close to $p_0$. In particular, the integer $b = \max\{|\alpha^k|\}$ can be chosen to be the same for the point $p$ as for $p_0$.

Also, for any multi-indices $\beta^1$ and $\eta$ such that

$$\frac{\partial^{|\eta|+|\beta^1|}}{\partial \chi^\eta \partial \tau^{\beta^1}}\bigg|_{\chi=0,\,\tau=0} \det\left(\frac{\partial {Q'_{H_0(p)}}^{r_k}_{\alpha^k}}{\partial \chi^j}(\chi, \tau)\right)_{k,j} \neq 0 \tag{6.28}$$

for the point $p = p_0$, we have that the same condition is true for all points $p \in M$ sufficiently close to $p_0$. Thus, we can choose the integer $B_1 = |\eta| + |\beta^1|$ and the multi-index $\beta^1$ to be the same for the point $p$ as for the point $p_0$. From Eq. (5.11), we then see that

$$N_k^1(p) \leqslant N_k^1(p_0). \tag{6.29}$$

We also have for any $j \geqslant 1$ that the coefficients of the vector-valued power series $S^j(z, \chi, z^2, \chi^2, \ldots; t)$ depend continuously on the point $p$ in a neighborhood of $p_0$. Therefore we also get, in a similar way as before, that

$$q_j(p) \leqslant q_j(p_0), \quad 2 \leqslant j \leqslant d+1. \tag{6.30}$$

We also see, in the same way as before, that if $p$ is sufficiently close to $p_0$, then we can choose $B_j$, for $2 \leqslant j \leqslant d+1$ and $\beta^j$ for $2 \leqslant j \leqslant d+1$ to be the same for $p$ as for $p_0$. From the recursion formula (6.25), we see that

$$N_k^j(p) \leqslant N_k^j(p_0), \quad 1 \leqslant j \leqslant d+1,\ k \geqslant 0. \tag{6.31}$$

In particular, we have that $N_0^{d+1}(p) \leqslant N_0^{d+1}(p_0)$.

Let $N(p)$ be the minimum of $N_0^{d+1}(p)$ over all choices of normal coordinates of $M$ at $p$ and $M'$ at $H_0(p)$, all choices of pairs $(\alpha^j, r_j)$, $1 \leqslant j \leqslant n$ satisfying condition (6.27), and all choices of pairs $(B_j, \beta^j)$, $1 \leqslant j \leqslant d+1$, where $B_j$ is an integer and $\beta^j$ a multi-index such that conditions (6.28) and (6.24) hold. Let $p_0 \in M$. For any choice realizing this minimum at $p_0$, from the observation above that $N_0^{d+1}(p) \leqslant N_0^{d+1}(p_0)$ when $p$ is sufficiently close to $p_0$ we have that $N(p) \leqslant N(p_0)$. So we have proved that the function $N(p)$ is upper semicontinuous. $\square$

We conclude this section, by showing how Corollary 1.5 follows from Theorem 1.4.

**Proof of Corollary 1.5.** Let $N(p)$ be the integer we get from Theorem 1.4 when $M' = M$ and $H_0$ is the identity mapping. Because we assume $N(p) \geqslant 0$, we assume that $H_1(p) = H_2(p)$ so if we set $H = H_1 \circ H_2^{-1}$, we see that the conclusion follows from the theorem. □

## 7. The infinite type case

In this section, we will prove Theorem 1.2 in the case where the source hypersurface $M$ is of infinite type at $p$. We will work with local coordinates (formal parameterizations of the hypersurfaces $M$ and $M'$), and use the technique developed by Ebenfelt in [12] (for smooth CR-mappings). However, in this section, all mappings and hypersurfaces are formal.

As in the previous sections, we are going to assume that $(M, p)$ and $(M', p')$ are given in normal coordinates, and work with formal mappings $H = (F, G)$ satisfying the condition

$$\det H_Z(Z) \not\equiv 0, \tag{7.1}$$

where $Z = (z, w)$. As pointed out in Section 2 (Lemma 2.1), from the fact that $H$ is a mapping between two hypersurfaces, it follows from (7.1) that

$$\det \frac{\partial}{\partial z} F\big(z, Q(z, \chi, \tau)\big) \not\equiv 0. \tag{7.2}$$

We know that $M$ is not Levi flat from the assumption that $M'$ is holomorphically nondegenerate together with (7.1).

Let $M$ be defined formally by

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w). \tag{7.3}$$

The assumption that $M$ is of infinite type means that $\phi(z, \bar{z}, 0) \equiv 0$.

We will parameterize $M$ by $z$, $\bar{z}$ and the real part of $w$, which we will denote by $s$, and we will use a similar parameterization on $M'$. We define the parameterized form of our mapping as

$$f(z, \bar{z}, s) = F\big(z, s + i\phi(z, \bar{z}, s)\big), \tag{7.4}$$

$$\hat{s}(z, \bar{z}, s) = \frac{G(z, s + i\phi(z, \bar{z}, s)) + \bar{G}(\bar{z}, s - i\phi(z, \bar{z}, s))}{2}, \tag{7.5}$$

and we let $h : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n \times \mathbb{R}$ be the formal (parameterized) mapping given by $h = (f, \hat{s})$.

A basis for the vector fields on $M$ in local coordinates is given by $T$, $L_A$, and $L_{\bar{A}}$, where

$$T = \frac{\partial}{\partial s}, \tag{7.6}$$

$$L_A = \frac{\partial}{\partial z^A} + \frac{i\phi_{z^A}}{1 - i\phi_s} \frac{\partial}{\partial s}, \tag{7.7}$$

$$L_{\bar{A}} = \frac{\partial}{\partial \bar{z}^A} - \frac{i\phi_{\bar{z}^A}}{1 + i\phi_s} \frac{\partial}{\partial s}, \tag{7.8}$$

where the index $A$ runs from 1 to $n$.

We are going to use the summation convention, so whenever an index is appearing both up and down in a term, we are going to take the sum over this index.

By $\theta$, $\theta^A$ and $\theta^{\bar{A}}$, we denote the dual basis, i.e. covectors satisfying

$$\langle\theta; T\rangle = 1, \qquad \langle\theta; L_B\rangle = 0, \qquad \langle\theta; L_{\bar{B}}\rangle = 0, \tag{7.9}$$

$$\langle\theta^A; T\rangle = 0, \qquad \langle\theta^A; L_B\rangle = \delta^A_B, \qquad \langle\theta^A; L_{\bar{B}}\rangle = 0, \tag{7.10}$$

$$\langle\theta^{\bar{A}}; T\rangle = 0, \qquad \langle\theta^{\bar{A}}; L_B\rangle = 0, \qquad \langle\theta^{\bar{A}}; L_{\bar{B}}\rangle = \delta^A_B. \tag{7.11}$$

Explicitly, we have

$$\theta = -\frac{i\phi_{z^A}}{1 - i\phi_s}\,dz^A + \frac{i\phi_{\bar{z}^A}}{1 + i\phi_s}\,d\bar{z}^A + ds, \tag{7.12}$$

$$\theta^A = dz^A, \tag{7.13}$$

$$\theta^{\bar{A}} = d\bar{z}^A. \tag{7.14}$$

The condition (7.2) is equivalent to

$$\det\big(L_A f^B\big)_{B,A} \not\equiv 0. \tag{7.15}$$

As in [12],

$$h_{\bar{A}_1\ldots\bar{A}_k B} := \langle\mathcal{L}_{\bar{A}_k}\ldots\mathcal{L}_{\bar{A}_1}\theta; L_B\rangle, \tag{7.16}$$

where $\mathcal{L}_{\bar{A}}$ is the Lie-derivative along the vector field $L_{\bar{A}}$. That is, given a one-form $\omega$, we have

$$\mathcal{L}_{\bar{A}}\omega = L_{\bar{A}} \lrcorner\, d\omega + d(L_{\bar{A}} \lrcorner\, \omega). \tag{7.17}$$

However, if $\omega$ is a holomorphic form then the second term vanishes and $\mathcal{L}_{\bar{A}}\omega$ is also a holomorphic form. That is, in the definition of $h_{\bar{A}_1\ldots\bar{A}_k B}$, we have $\mathcal{L}_{\bar{A}}\omega = L_{\bar{A}} \lrcorner\, d\omega$.

Similarly, we define

$$h_{\bar{A}_1\ldots\bar{A}_k} := \langle\mathcal{L}_{\bar{A}_k}\ldots\mathcal{L}_{\bar{A}_1}\theta; T\rangle. \tag{7.18}$$

From [12], we have the following recursive relation:

$$h_{\bar{A}_1\ldots\bar{A}_k\bar{C}B} = L_{\bar{C}} h_{\bar{A}_1\ldots\bar{A}_k B} + h_{\bar{A}_1\ldots\bar{A}_k} h_{\bar{C}B}. \tag{7.19}$$

One can easily check using this relation that $h_{\bar{A}_1\ldots\bar{A}_k B}$ is symmetric in the first $k$ indices. Therefore we can instead use the multi-index notation

$$h_{\bar{\alpha}B} := h_{\underbrace{\bar{1}\ldots\bar{1}}_{\alpha_1}\underbrace{\bar{2}\ldots\bar{2}}_{\alpha_2}\ldots\underbrace{\bar{n}\ldots\bar{n}}_{\alpha_n}B}, \tag{7.20}$$

because the ordering of the indices is unimportant.

On our target manifold, we have a similar parameterization, basis, dual basis, and functions. We denote each of these by putting a hat over the corresponding object.

It is known that $M'$ is (formally) holomorphically nondegenerate if and only if there exist multi-indices $\alpha^1, \ldots, \alpha^n$ such that

$$\det(\hat{h}_{\bar{\alpha^j} k})_{j,k} \not\equiv 0. \tag{7.21}$$

Because $h$ is a formal CR-map, $h_*$ of any formal CR-vector on $M$ is a formal CR-vector on $M'$, so there exist power series $\gamma_A^B = \gamma_A^B(z, \bar{z}, s)$ such that $h_*(L_{\bar{A}}) = \bar{\gamma}_A^B \hat{L}_{\bar{B}}$. This implies that $h_*(L_A) = \gamma_A^B \hat{L}_B$. We let the power series $\xi = \xi(z, \bar{z}, s)$ and $\eta = \eta(z, \bar{z}, s)$ be defined by $\xi = \langle \hat{\theta}, h_* T\rangle$ and $\eta^A = \langle \hat{\theta}^A, h_* T\rangle$. Observe that we have $h_* T = \xi \hat{T} + \eta^A \hat{L}_A + \overline{\eta^A} \hat{L}_{\bar{A}}$. We summarize this in the matrix notation (still using the summation convention)

$$h_* \begin{pmatrix} T \\ L_A \\ L_{\bar{A}} \end{pmatrix} = \begin{pmatrix} \xi & \eta^B & \overline{\eta^B} \\ 0 & \gamma_A^B & 0 \\ 0 & 0 & \overline{\gamma_A^B} \end{pmatrix} \begin{pmatrix} \hat{T} \\ \hat{L}_B \\ \hat{L}_{\bar{B}} \end{pmatrix}. \tag{7.22}$$

By duality, we then have

$$h^* \begin{pmatrix} \hat{\theta} \\ \hat{\theta}^B \\ \hat{\theta}^{\bar{B}} \end{pmatrix} = \begin{pmatrix} \xi & 0 & 0 \\ \eta^B & \gamma_A^B & 0 \\ \overline{\eta^B} & 0 & \overline{\gamma_A^B} \end{pmatrix} \begin{pmatrix} \theta \\ \theta^A \\ \theta^{\bar{A}} \end{pmatrix}. \tag{7.23}$$

It is straightforward to derive the following equalities, see [12]:

$$\xi h_{\bar{A}B} = \overline{\gamma_A^C} \gamma_B^D \hat{h}_{\bar{C}D}, \tag{7.24}$$

$$L_{\bar{A}}\xi + \xi h_{\bar{A}} = \xi \overline{\gamma_A^C} \hat{h}_{\bar{C}} + \overline{\gamma_A^C} \eta^D \hat{h}_{\bar{C}D}, \tag{7.25}$$

$$L_{\bar{A}} \gamma_B^E + \eta^E h_{\bar{A}B} = 0, \tag{7.26}$$

$$L_{\bar{A}} \eta^C + \eta^C h_{\bar{A}} = 0, \tag{7.27}$$

$$T \gamma_A^C - L_A \eta^C - \eta^C \overline{h_{\bar{A}}} = 0. \tag{7.28}$$

We will prove some relations among $\xi$, $\gamma$ and $\eta$, which will help us to later get a system of differential equations.

We have

$$L_A f^B = L_A \hat{z}^B = \gamma_A^C \hat{L}_C \hat{z}^B = \gamma_A^B, \tag{7.29}$$

$$T f^A = \left(\xi \hat{T} + \eta^B \hat{L}_B + \overline{\eta^B} \hat{L}_{\bar{B}}\right) f^A = \eta^A \tag{7.30}$$

and

$$T \hat{s} = \left(\xi \hat{T} + \eta^B \hat{L}_B + \overline{\eta^B} \hat{L}_{\bar{B}}\right) \hat{s} = \xi + \eta^B \frac{i \hat{\phi}_{z^B}}{1 - i \hat{\phi}_s} - \overline{\eta^B} \frac{i \hat{\phi}_{\bar{z}^B}}{1 + i \hat{\phi}_s}. \tag{7.31}$$

Thus, the assumption (7.15) gives us that

$$\det\left(\gamma_B^A\right)_{A,B} \not\equiv 0. \tag{7.32}$$

Let $\gamma$ denote the matrix $(\gamma_B^A)_{A,B}$ and let $\gamma^*$ be the classical adjoint of the matrix $\gamma$. That is,

$$\gamma^{*A}_{\ B}\gamma_C^B = \gamma^{*B}_{\ C}\gamma_B^A = \det(\gamma)\cdot\delta_C^A, \tag{7.33}$$

where $\delta_C^A$ is the Dirac delta symbol.

From (7.24) and (7.25), we get

$$\gamma_B^D\hat{h}_{\bar{A}D} = \frac{\overline{\gamma^{*C}_{\ A}}h_{\bar{C}B}\xi}{\det(\bar{\gamma})}, \tag{7.34}$$

$$\eta^D\hat{h}_{\bar{A}D} = \frac{\overline{\gamma^{*C}_{\ A}}(L_{\bar{C}}\xi + \xi h_{\bar{C}}) - \xi\hat{h}_{\bar{C}}\cdot\det(\bar{\gamma})}{\det(\bar{\gamma})}. \tag{7.35}$$

We can now prove the following lemma:

**Lemma 7.1.** *For any integer $k \geqslant 1$ and indices $A_1,\dots,A_k,B \in \{1,\dots,n\}$ the following identity holds*

$$\gamma_B^D\hat{h}_{\bar{A}_1\dots\bar{A}_kD} = \frac{r_{\bar{A}_1\dots\bar{A}_kB}(\overline{L^J\gamma_A^C}, L^{\bar{I}}\xi; h)}{\det(\bar{\gamma})^{2k-1}} \tag{7.36}$$

*and for any indices $A_1,\dots,A_k \in \{1,\dots,n\}$ the following identity holds*

$$\eta^D\hat{h}_{\bar{A}_1\dots\bar{A}_kD} = \frac{s_{\bar{A}_1\dots\bar{A}_k}(\overline{L^J\gamma_A^C}, L^{\bar{I}}\xi; h)}{\det(\bar{\gamma})^{2k-1}}, \tag{7.37}$$

*where*

$$r_{A_1,\dots,A_k,B}\big(\overline{L^J\gamma_A^C}, L^{\bar{I}}\xi; \hat{z}', \bar{z}', \hat{s}'\big)(z,\bar{z},s) \quad \textit{and}$$
$$s_{A_1,\dots,A_k}\big(\overline{L^J\gamma_A^C}, L^{\bar{I}}\xi; z', \bar{z}', s'\big)(z,\bar{z},s) \tag{7.38}$$

*are power series which are polynomials in the arguments proceeding the ";", and where $A$, $C$ run over the indices $\{1,\dots,n\}$ and where $I$ and $J$ run over all multi-indices with $|I|,|J| \leqslant k$. Moreover the functions in* (7.38) *depend only on $M$ and $M'$ (and not on the mapping $h$).*

**Proof.** From (7.34) and (7.35), we see that the lemma is true for $k=1$. By assuming that the lemma holds for a fixed $k$ and applying the vector field $\frac{\overline{\gamma^{*C}_{\ A_{k+1}}}}{\det(\bar{\gamma})}L_{\bar{C}}$ to both sides of (7.36) and (7.37), we see that the lemma holds for $k+1$. □

Because $M'$ is holomorphically nondegenerate, there exist multi-indices $\alpha^1,\dots,\alpha^n$, where $\alpha^A = (\alpha_1^A,\dots,\alpha_n^A)$ such that $\det(\hat{h}_{\bar{\alpha}^A,B})_{A,B}(z',\bar{z}',s') \not\equiv 0$.

We define

$$\hat{\lambda}(z',\bar{z}',s') := \det(\hat{h}_{\bar{\alpha}^A,B})_{A,B}(z',\bar{z}',s'). \tag{7.39}$$

From (7.1), we get

$$\det\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \bar z} & \frac{\partial f}{\partial s} \\ \frac{\partial \bar f}{\partial z} & \frac{\partial \bar f}{\partial \bar z} & \frac{\partial \bar f}{\partial s} \\ \frac{\partial \hat s}{\partial z} & \frac{\partial \hat s}{\partial \bar z} & \frac{\partial \hat s}{\partial s} \end{pmatrix}(z,\bar z,s) \not\equiv 0. \tag{7.40}$$

Therefore,

$$\hat\lambda\big(f(z,\bar z,s),\, \bar f(\bar z,z,s),\, \hat s(z,\bar z,s)\big) \not\equiv 0. \tag{7.41}$$

From Lemma 7.1, we have, using the multi-index notation,

$$\gamma_B^D \hat h_{\bar\alpha^A D} = \frac{r_{\bar\alpha^A B}(\overline{L^J\gamma_A^C}, L^{\bar I}\xi; h)}{\det(\bar\gamma)^{2k-1}} \tag{7.42}$$

and

$$\eta^D \hat h_{\bar\alpha^A D} = \frac{s_{\bar\alpha^A}(\overline{L^J\gamma_A^C}, L^{\bar I}\xi; h)}{\det(\bar\gamma)^{2k-1}}. \tag{7.43}$$

Let the matrix $(\hat h^{*\bar C D})_{\bar C,D}$ be the classical adjoint of the matrix $(\hat h_{\bar\alpha^A B})_{A,B}$. That is,

$$\hat h^{*\bar A B}\hat h_{\bar\alpha^A D} = \hat\lambda\cdot\delta_D^B, \tag{7.44}$$

$$\hat h^{*\bar A B}\hat h_{\bar\alpha^C B} = \hat\lambda\cdot\delta_{\bar C}^{\bar A}. \tag{7.45}$$

From the proof of Proposition 3.18 in [12], we have the following:

**Lemma 7.2.** *For any integer $k \geqslant 0$ and multi-index $J$, we have the following identity.*

$$\sum_{m=1}^{|J|+k}\sum_{s=0}^{k} b_s^{E_1,\ldots,E_n,E,\bar F}\underbrace{\big[\ldots\big[[L_{E_1}L_{E_2}\ldots L_{E_n}, L_{\bar F}], L_{\bar F}\big]\ldots, L_{\bar F}\big]}_{\text{length } s} = (h_{\bar F E})^p L^J T^k, \tag{7.46}$$

*where $p$ is an integer which depends on $k$, and $J$, and where $b_s^{E_1,\ldots,E_n,E,\bar F}$ are power series.*

Because $M$ is not Levi-flat, we can pick indices $E$ and $F$ such that $h_{\bar F E} \not\equiv 0$.

By following [12], we will define the classes $C_p^a$ and the refined classes $C_{p,q}^{a,b}$ as follows: We shall say for a power series on $M$ (i.e. a power series in $z,\bar z,s$) that $u \in C_{p,q}^{a,b}$ if

$$u = \frac{r(\overline{L^I\gamma_A^C}, \overline{L^I T^m\eta^C}, \overline{L^I T^m\xi}, L^N\gamma_A^C, L^N T^n\eta^C; h)}{(\det\bar\gamma)^{j_1}(h_{\bar F E})^{j_2}(\hat\lambda\circ h)^{j_3}}, \tag{7.47}$$

for some power series $r$ and nonnegative integers $j_k$ for $1 \leqslant k \leqslant 6$, where the power series $r$ is a polynomial in the arguments preceding the ";". The indices $A$, $C$ run over the set $\{1,\ldots,n\}$. The multi-indices $I$, $N$, and nonnegative integers $m$, $n$ run over $|I| + m \leqslant p$, $m \leqslant q$, $|N| + n \leqslant a$, $n \leqslant b$. Moreover, $r$ should only depend on $M$, $M'$, but not the mapping $h$.

**Remark 7.3.** From (7.41), we have that $\hat{\lambda} \circ h \not\equiv 0$, so the right hand side of (7.47) is in the quotient ring of formal power series, but the condition that $u$ is a power series says that the right hand side factors. In other words (7.47) means that the following power-series identity holds:

$$u \cdot (\det \bar{\gamma})^{j_1} (h_{\bar{F}E})^{j_2} (\hat{\lambda} \circ h)^{j_3} = r\big(\overline{L^I \gamma_A^C}, \overline{L^I T^m \eta^C}, \overline{L^I T^m \xi}, L^N \gamma_A^C, L^N T^n \eta^C; h\big). \tag{7.48}$$

Let $k_0 = \max_A |\alpha^A|$. By multiplying (7.42) and (7.43) with the classical adjoint and dividing with the determinant, we get

$$\gamma_F^D, \eta^C \in C_{k_0,0}^{-1,-1}. \tag{7.49}$$

From the reality of $\xi$, we also have directly that

$$\xi \in C_{k_0,0}^{-1,-1}. \tag{7.50}$$

Now, the same arguments as in [12] (with $l_0 = 1$ and with the classes $D_{p,q}^{a,b}$ having quotient similar as the classes $C_{p,q}^{a,b}$ defined above) give us that

$$L^J T^k \gamma_D^B, L^J T^k \eta^B \in C_{k_0+|J|+k, \min(k_0, |J|+k)}^{-1,-1} \tag{7.51}$$

and

$$L^J T^k \gamma_D^B, L^J T^k \eta^B, L^J T^k \xi \in C_{3k_0,3k_0}^{-1,-1}, \quad \forall k \leqslant k_0. \tag{7.52}$$

By using (7.52) on the conjugated terms in (7.51), Lemma 7.2, and Eqs. (7.25) through (7.27), we finally get the following:

**Lemma 7.4.** *For any multi-indices $R$ and $Q$, any nonnegative integer $k$, and indices $D$ and $F$, there are power series $r_1$, $r_2$ and $r_3$, which are polynomials in their arguments preceding the ";", and monomials $m_1$, $m_2$ and $m_3$ such that*

$$L^R T^k L^{\bar{Q}} \gamma_F^D = \frac{r_1(L^I \gamma_A^C, L^I T^j \eta^C, L^I T^j \xi; h)}{m_1(\det \gamma, \det \bar{\gamma}, h_{\bar{F}E}, \overline{h_{\bar{F}E}}, \hat{\lambda} \circ h, \overline{\hat{\lambda} \circ h})}, \tag{7.53}$$

$$L^R T^k L^{\bar{Q}} \eta^D = \frac{r_2(L^I \gamma_A^C, L^I T^j \eta^C, L^I T^j \xi; h)}{m_2(\det \gamma, \det \bar{\gamma}, h_{\bar{F}E}, \overline{h_{\bar{F}E}}, \hat{\lambda} \circ h, \overline{\hat{\lambda} \circ h})}, \tag{7.54}$$

$$L^R T^k L^{\bar{Q}} \xi = \frac{r_3(L^I \gamma_A^C, L^I T^j \eta^C, L^I T^j \xi, \overline{L^I \gamma_A^C}, \overline{L^I T^j \eta^C}, \overline{L^I T^j \xi}; h)}{m_3(\det \gamma, \det \bar{\gamma}, h_{\bar{F}E}, \overline{h_{\bar{F}E}}, \hat{\lambda} \circ h, \overline{\hat{\lambda} \circ h})}, \tag{7.55}$$

*where $|I| + j \leqslant 3k_0$ and where the functions $r_1$, $r_2$, $r_3$, $m_1$, $m_2$ and $m_3$ only depends on the hypersurfaces $M$ and $M'$ and not on the mapping $h$.*

We will use (7.53), (7.54), the complex conjugates of (7.53) and (7.54) for $|I| + j = 3k_0$ to obtain that $T(L^I \gamma_A^C)$, $T(L^I T^j \eta^C)$, $T(L^I T^j \xi)$, $T(\overline{L^I \gamma_A^C})$, $T(\overline{L^I T^j \eta^C})$ and $T(\overline{L^I T^j \xi})$ can all be expressed in terms of lower order derivatives and with a quotient which is a monomial in $\det \gamma$, $\det \bar{\gamma}$, $h_{\bar{F}E}$, $\hat{\lambda} \circ h$ and $\overline{\hat{\lambda} \circ h}$. After putting these expressions on a common denominator, the denominator will be

$$(\det \gamma)^{j_1} (\det \bar{\gamma})^{j_2} (h_{\bar{F}E})^{j_3} (\overline{h_{\bar{F}E}})^{j_4} (\hat{\lambda} \circ h)^{j_5} (\overline{\hat{\lambda} \circ h})^{j_6} \tag{7.56}$$

for some integers $j_k$, $1 \leqslant k \leqslant 6$.

Let $U$ be the vector $(f_A, \bar{f}_A, \hat{s}, L^I \gamma_A^C, L^I T^j \eta^C, L^I T^j \xi, \overline{L^I \gamma_A^C}, \overline{L^I T^j \eta^C}, \overline{L^I T^j \xi})$, where $A$, $C$ run over all indices and $I$, $j \geqslant 0$ runs over the set $|I| + j \leqslant 3k_0$. From the argument above and Eqs. (7.29) through (7.31), we see that we can express $TU$ in terms of $U$. We obtain a singular system of ode's:

$$\partial_s U(z, \bar{z}, s) = \frac{R(z, \bar{z}, s, U)}{q(U)}, \tag{7.57}$$

where $q(U)$ is the common denominator (7.56). Note that we have $q(U(z, \bar{z}, s)) \not\equiv 0$.

We can now prove Theorem 1.2 in the case where $M$ is of infinite type at $p$. Let $U_0$ be the vector-valued power series corresponding to the mapping $H_0$. From the assumption that $M$ is of infinite type, we get from Proposition 5.1 that for any $k$ there is an $N_k$ such that if $H$ is any mapping with $j^{N_k}(H) = j^{N_k}(H_0)$ and $U$ is the vector-valued power series defined above corresponding to $H$, we have that $\frac{\partial^j U}{\partial s^j}(z, \bar{z}, 0) = \frac{\partial^j U_0}{\partial s^j}(z, \bar{z}, 0)$ for $j \leqslant k$. The conclusion of the theorem now follows from Theorem 3.2 after centering the vector-valued power series $U$ at 0. That is, applying the theorem to the vector $\tilde{U}(z, \bar{z}, s) = U(z, \bar{z}, s) - U(0)$.

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